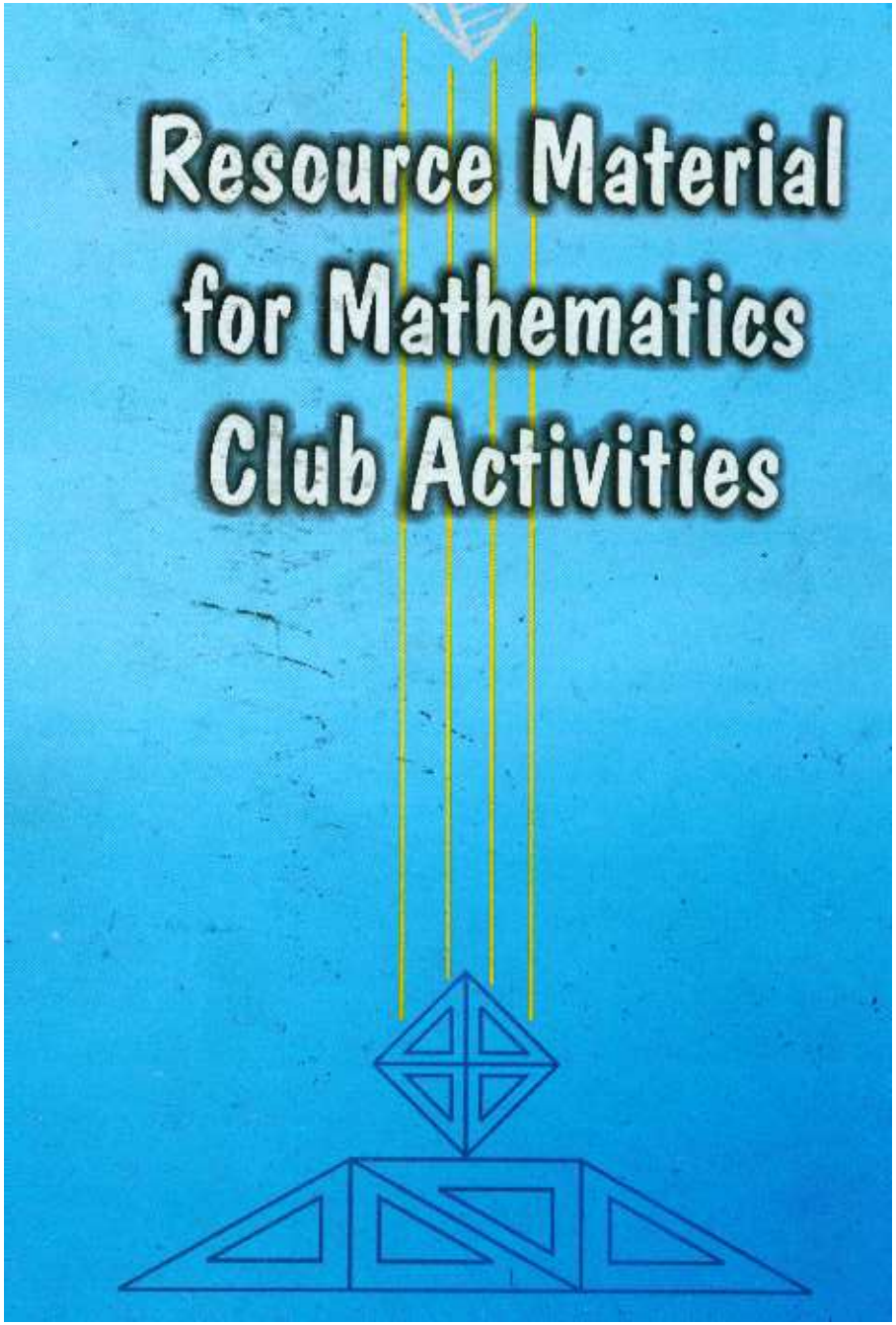

Resource Material
for Mathematics
Club Activities

About the Book

Motivating children to learn mathematics with interest and involvement through appreciation of its intrinsic worth poses a challenge to practising teachers of mathematics. One of the ways of solving this crucial problem is to expose children to mathematics in the environment and help them to engage themselves in manipulating ordinary objects and numbers associated with them so as to experience the mathematics inherent in such manipulations without resorting to a formal study of mathematics.

Shri P. K. Srinivasan is a committed mathematics educator with a rich experience gained through conducting more than fifty mathematics expositions by children in India and abroad. Through his articles appearing in the 'Science and Children' columns of *The Hindu* since 1979, he has been consistently showing how this motivation can be acquired delightfully and successfully.

With the objective of disseminating the strategies suggested by the author on a wider scale, a compilation of his articles suitably arranged, modified and edited has been brought out by the NCERT. Each topic is complete in itself by way of the message it conveys, as a result of which any topic can be taken up in the book without difficulty. Special efforts have been made to see to it that the background knowledge needed to understand any topic requires only the rudiments of mathematics acquired in the pre-high school classes.

Once the teachers and parents implement the strategies outlined in these articles, children will find themselves discovering mathematics on their own, and that will develop in them a lifelong aptitude for mathematics.

The presentation and treatment of topics provide a rich source of ideas for running mathematics clubs and mathematics fairs, which have become the hallmark of progressive schools in mathematics education today.

The book does not reflect the views of the NCERT. The Council is just making available one more source of motivational aids for mathematics learning.

Dr. Harmesh Lal has compiled the articles and got the illustrations finalised for printing.

Contents

FOREWORD

A. ACTIVITIES FOR BASIC CONCEPTS

1.	Number Naming Strategy	1
2.	Multiplication Facts with Broom Sticks	7
3.	Discovering Number Properties - I	11
4.	Discovering Number Properties - II	17
5.	Games to Learn Integers	21
6.	Surprises with Clock Arithmetic	26
7.	New Wine in Old Bottles	31
8.	An Integral View of Mathematics	38
9.	Mathematics with Two Graduated Rulers 'Scales'	42
10.	Visual Aids for Multiplication Process	46
11.	Instil Number Sense in Tiny Tots	49
12.	Conversion of Numeral Base on Fingers	52
13.	Number Formations	56
14.	Function Concept Game - I	59
15.	Function Concept Game - II	61

B. EXPERIENCES WITH SHAPES

1.	Maths with Railway Tickets	64
2.	Not a Game of Chance	68
3.	Maths with a Ruled Sheet	72
4.	Explorations on a Ruled Sheet	76
5.	Checkruled Areas	81
6.	Fun with Cardboard Shapes	85
7.	The Nets that Make Up a Box	91
8.	Shapes with Set Squares	94
9.	Instant Construction of Solid Shapes	97

C. FUN WITH NUMBERS

1.	Fun Time with Calendars - I	99
2.	Fun Time with Calendars - II	105
3.	Magic Squares for Greetings	110
4.	Altering the Magic Square with Minimum Chances	114
5.	Currency Notes and Fibonacci Numbers	117

6. Some Fascinating Properties of Fibonacci Numbers 120
7. Number in Kolam 123

D. GAMES FOR ENRICHMENT

1. Bowls Set and Transfer Game 127
2. A Game with Playing Cards 132
3. 'Guess the Number Games 137
4. A Game of Number Prediction 141
5. Discover Fun with Bus Tickets 145
6. Mathematics with Match Boxes 149
7. An Inverting Game with Coins 153
8. Number Games 158

E. INVESTIGATION (Non-routine problems)

1. Time for Investigation 161
2. Prods to Mathematical Thinking 165
3. Investigational Spree 168
4. Narayana Pandita Game of Moves 172
5. A Mathematics Project with Fruits 177
6. Exciting Geometric Connections 181

A. Activities for Basic Concepts

A1 – Number Naming Strategy

1. Prologue

One of the most abstract concepts in mathematics is the counting of a number, for the simple reason that the number is *independent of colour, shape, space, order, time, mass, volume* etc. of the objects considered for counting. Is it not a marvel that a child can grapple with this concept with such extraordinary ease? That a child can do it is a tribute to its power to understand and learn abstract concepts.

Given any collection of objects, concrete or abstract, one can always find physically or mentally as many collections as one may desire, so that objects in any of the new collections can be put into *object to object*, or, *one to one* matching with objects of the given collection. This situation generates the idea of number and starts the naming spree.

Numerous systems of naming numbers emerged, as a result of which, a grave problem arose. All the words in all the languages are not enough to name all possible numbers, that is if we want to use separate words for them. This is a challenging situation but the one that is well within the understanding capacity of a child. 'Tell me a number as big as you like; I can always tell you a bigger number by adding one more to yours' and this challenge brings out intuitively the fact that there are countless numbers.

This problem of naming was such a serious and complex one that it took centuries for man to evolve the concept of units and higher units on the prior acceptance of a base and make the solution incredibly simple.

What is base ten? It is simply counting in tens, forming tens and tens of tens etc. as higher units; counting the higher units in tens and forming still higher units and so on. Take a collection of objects. Count them in *tens* (by matching them if

need be with the fingers of your hands); you get tens and a few objects less than ten (that may or may not be there). Now tens form higher units. Count these higher units in tens if possible. You get *ten tens* or *hundreds* and a few tens less than ten tens (that may or may not be there). Now ten tens or hundreds form still higher units. Count these ten tens or hundreds in tens if possible. You get *ten (ten tens)* or *ten hundreds* or *thousands* and a few ten tens or hundreds less than ten (ten tens) (that may or may not be there).

This process of counting is continued giving rise to a succession of higher units. The process will end if a collection of physical objects is taken, as the collection is bound to be a finite one. Conceptually, one can continue the process non-stop. Every time you form still higher units out of higher units obtained, you count them in tens and hence *ten* is the basis in this counting system.

A question arises: 'Why should we count in tens ?' History reveals that other bases such as two, five, twelve, twenty and sixty have been tried. But ten has come to stay as the universally accepted base for the simple reason that it tallies with the number of fingers, one of the characteristics of the human race. Also it is neither too small nor too big.

It took a few more centuries to devise positional notation which became perfect with the introduction of zero. It is interesting to note that *zero was invented first as a place holder and discovered later as a number in its own right.* Since the positional notation holds for any base, one should be careful in reading a numeral.

2. Meaning of symbol 100

For instance ask anyone to read 100. He would read it readily as one hundred, *assuming* the base to be ten. The better way to read it regardless of any base is *one zero zero.* If the base is two, 100 stands for *two twos* or *four;* if the base is five, 100 stands for *five fives* or *twenty five.* That is the intriguing and at the same time striking thing about positional notation. *Writing numerals for a number is easier than naming it verbally.*

Helping a child to relive the experiences of counting in different bases in stages and write a number in different numerals is therefore of great educational value.

3. The Strategy

Collect forty tins (coconut 'half' shells would be handy) and about three hundred small seeds and about thirty broomstick bits. Ask the child to place the tins in columns of ten to start with, leaving equal gaps between columns. The tins can be close to each other as in each column (Fig. 1).

The column towards the extreme right would represent *units.* The column next to it on the left or the second column would represent *tens.* The third column would represent *hundreds.* This will become abundantly clear and meaningful once the child is engaged in the activity explained below:

Take a large collection of seeds for counting. Put one seed in each of the tins in the *first* column starting with the bottom most tin. When all the tins in the column have a seed each, collect all the seeds and put them together in the *first tin* (i.e. bottom most) of *column two.* Start again putting one seed in each of the tins in column *one.* As soon as all the tins have a seed each, collect all the seeds and put them together in the *second tin of column two.*

Continue this process. Every tin in the second column will be getting ten seeds. When each one of the tins in the second column have each received ten seeds, collect all the seeds in the tins of column two and put them all together in the first tin of column three. This is the pattern of activity to be kept up till all the seeds taken for counting is as shown in Fig. 2. The count of the seeds is made up of *two hundreds, four tens* and *two ones, that is* 242.

The experience of tediousness in collecting all the seeds each time a column of tins is filled up and facing the problem of finding space for seeds to be put into tins in higher columns suggests *a strategy.* Once the first column is filled up, empty the tins and put a *broomstick bit* in the first tin (bottom most) of the second column to represent ten seeds. *The idea of representation is a great leap forward in conceptual development.*

Resume filling the tins of the first column at the rate of one seed per tin and when all the tins are filled up, remove all the seeds and put one broomstick bit in the second tin of the second column. Continue the process and if the tins of the second column have each received a broomstick bit, empty the tins of the second column and put one bit in the first tin of the third column. Now this bit represents *ten tens* or *a hundred.* Continue this improved process as long as is required.

Now comes yet another improvement in the strategy. Seeds and counters (broomstick bits here) are not mixed up. Each seed, each collection of hundred seeds and so on will now be shown by representation through counters put appropriately in the tins (Fig. 3). The great advantage in this process is that all the seeds to be counted are collected back again and their count is read off the columns of tins by inspection and this paves the way for the use of *spike abacus with spikes representing place values.* Also the same collection is used to find its number expressed in different numerals, with the change of base.

Once the child masters recognition of *digital numerals 0 to* 9, it is ready to enjoy finding numerals to different bases less than ten for the same number of objects. *For bases more than ten, new digit symbols have to be contrived.* For example, for base twelve, two more digits besides 0 to 9 are needed to represent *ten* and *eleven* as *twelve will now be represented by the numeral 10,* meaning *one twelve* and *zero units.*

Remove the topmost or the tenth row. You can count now to base nine numeration by a process similar to the one described earlier and the digits you will need are *0 to* 8 as *10 would mean now nine.* Remove the top two rows or the tenth and the ninth rows. You can count now to base eight with digits *0 to 7 as 10 would mean now eight.* And so on. By placing one more row at the top of the ten rows, you can count to base eleven with digits *0 to* as*10 would mean eleven now.* By placing two more rows at the top of the ten rows, you can count not to base twelve with digits *0 to* as *10 would mean now twelve.* And so on (Fig. 4).

This strategy of having columns of tins with uniform change in the number of tins in each column helps a child to experience naming a number in positional notation to different bases. Twenty one objects counted in base five, four, three and two are as shown in Fig. 5.

This game of finding different numerals for the same number brings home to the child by way of playing, the *difference between number and numeral* and *the positional role of digits.* With the number of digits the same as the base, one is thrilled to realise that one can name any number however, large it may be, by repetition of digits and assignment of place value. This can be presented briefly as shown in Fig. 4.

This game deserves to be played in every home, especially for children below eight years, so that they could be placed in a vantage position to learn with ease and understanding the operations of addition and subtraction, multiplication and division in the realm of whole numbers, Of course the child will have to spend more time in base ten numeration, as it will be the mainstay, in school mathematics.

No child will fail to see the meaning of symbols 10, 100, 1000, etc. as related to base or bases of counting. Some children even enjoy making symbols of digits in large base systems just for the fun of it; in one instance, for *base hundred* the hundred digits were made by combining digits of numerals 1 to 99 as shown below:

0, 1, 2, 3, ...9, 10, 11 .....20, 21, 22 .....so on the last digit being 99. All these compound symbols are taken as single digits in base hundred.

Number Naming Strategy

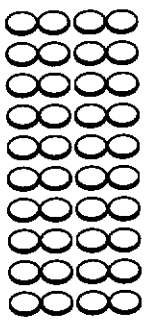


Fig. 1

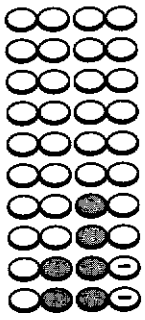


Fig. 2

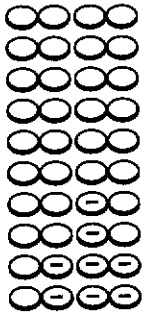


Fig. 3

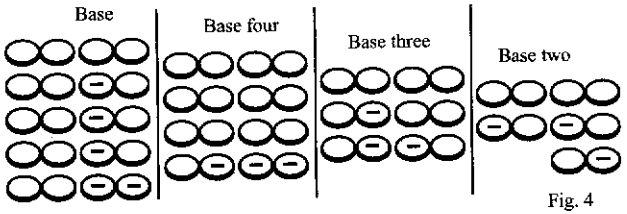
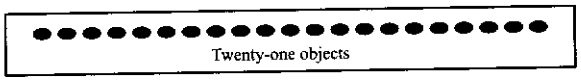


Fig. 4



41	=	111	=	210	=	10101
Base ten		Base five		Base three		Base two

Base ten	Base five	Base three	Base two
etc. Hundreds (10x10's) Tens(10's) Units(1's)	etc. Five fives (10x10's) or twenty fives Fives (10's) Units (1's)	etc. Three three's (10x10's) or nines Three's (10's) units (1's)	etc. Two two's (10x10's) or fours Two's (10's) Units (1's)
0 0 0 1 1 1 2 2 2 3 3 3 4 4 4 5 5 5 6 6 6 7 7 7 8 8 8 9 9 9	0 0 0 1 1 1 2 2 2 3 3 3 4 4 4	0 0 0 1 1 1 2 2 2	0 0 1 1
41 base five: read four one to base five (4 fives and one)	111 base four read one one one to base four (1 sixteen 1 four and one)	210 base three read two one zero to base three (2 nines 1 three and 0 units)	10101 base two read one zero one zero one to base two (1 sixteen 1 four and 1 one)

Fig. 5

A2 – Multiplication Facts with Broomsticks

1. Prologue

If children are encouraged to learn mathematics through their own efforts and experience under the supervision of a dedicated teacher, they develop a feel for what they learn and are able to find their way confidently in the world of mathematical thinking. But to develop it, a teacher has to avoid spoon feeding his students all the time and discourage them from learning by mere repetition.

Learning tables by rote carries with it a long and hoary tradition. Teachers often encourage it and parents underscore it as the latter have themselves gone through the mill in their own school days. With the advent of calculators, it would be ludicrous to insist on rote learning, when children can easily learn how to build the tables meaningfully and remember and recall basic multiplication facts through adequate familiarisation, just as they learn a language with natural ease. The only thing that is needed is a stimulating environment.

Since the *greatest basic multiplication fact is 9x9=81*, what is needed first is a set of eighteen (9+9=18) almost straight broomsticks which are available in all homes. Every lower primary child can even be asked to use them as a kit in class as well as at home, particularly while building and learning the tables. (If more sophisticated materials are to be preferred for durability and attraction; plastic tongue cleaners may be used) (Fig. 1).

The exercise starts with a few preliminary experiences of discovery by children. Two broomsticks can be placed only in two ways: (1) meeting each other at a point, and (2) parallel to each other. These represent the *incidence properties of two lines on a plane*. With more than two broomsticks, children can easily place them in such a way that any pair of them are parallel to each other and they can use another stick to meet

all of them by placing it across and count the meeting points.

Multiplication is as usual introduced as repeated addition by taking groups of equal number of objects. To read the multiplication facts fast, children should have acquired a fairly good mastery of *skip counting by twos, threes,* etc.

Skip counting is easily picked up by children when they write numbers in a zig zag way in *two rows for skip counting by twos, in three rows for skip counting by threes* and so on as illustrated below:

Skip counting by twos: 1 3 5 7 9 11 13 15 . . .
2 4 6 8 10 12 14 16. . .
Skip counting by threes: 1 4 7 1 0 1 3 1 6 1 9. . .
2 5 8 11 14 17 20 . . .
3 6 9 12 15 18 21 . . .

Building multiplication tables by repeated addition can be done up to five in each of the first five tables. It requires handling 25 objects. Beyond this stage, it becomes tedious to get multiplication facts by grouping. Herein comes the appropriateness of the *crisis-cross technique* by using broomsticks.

2. Use of broomsticks

The use of broomsticks by every child helps the teachers to dispense with charts or blackboard work where *tables are presented as finished products, to be copied and memorised by the children.* This method will bring about a sea change in the relationship between the children and the teacher.

How children can be guided to build, say 7 times table, using broomsticks is illustrated in fig. 2 step by step, only seven ones, seven twos and seven threes are given, as after that the pattern would take care of the rest.

Children find this exercise very enjoyable as they *see distinctly the multiplier, the multiplicand and the product* in each multiplication fact, unlike in giving products through numbers of groups of equal number of objects. By resorting to the broomsticks set-up and spelling out the multiplication facts thus obtained, children escape the tedium rote learning of tables. The underlying mathematical principle is the

Cartesian product which is simply pairing of objects taken from two sets at the rate of one from each set. Given two blouses and three skirts, the number of pairs of a blouse and a skirt that can be considered is obviously 2×3=6, as each of the blouses can be paired with each of the three skirts. The experience that one gains in this criss-cross exercise is not of curio value but is of great significance, for it would recur quite often in their mathematical learning process as they advance in years.

Above all, the broomstick way of building multiplication tables, if one may call it so, *has decidedly great superiority over* that of the traditional way of grouping of equal number of objects. *Multiplication facts of zero which are usually taught by dictation can now be discovered by children through their feel for patterns, as well as, intuition or imagination.*

3. Zero

This presupposes that children understand and use zero properly. Children learn to say I have got *zero sisters* or I have *zero brothers* whenever they have no sisters or brothers. A teacher in preschool years should provide numerous opportunities almost daily for this usage of zero. *Teaching zero as nothing creates misconceptions and so needs to be avoided.* For example consider 2509. There is nothing between 2 and 5 but there is 0 between 5 and 9. What information does 0, therefore, give here? Zero in 2509 means that there are zero tens, or in other words there are no tens to count in the tens place.

Once this becomes a habit with children, they are ready to discover multiplication facts of zero the broomstick way. The sequence of steps that a teacher has to take to elicit from children the corresponding multiplication fact in each step is illustrated in figure 3. One can start with a multiplication fact of any two non-zero numbers. Let us start here with 4×3 (See Figs. 3 and 4).

4. Versatility broomsticks as aids

Broomsticks have their own versatility as learning-teaching aids in mathematics education. They can be used to study incidence properties of lines on a plane, angles and kinds of

angles, building polygonal shapes with special attention to quadrilaterals and their properties, partitioning of a plane shape, etc.

Multiplication Facts with Broomsticks

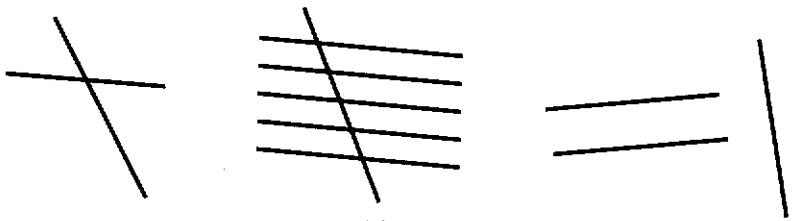


Fig. 1

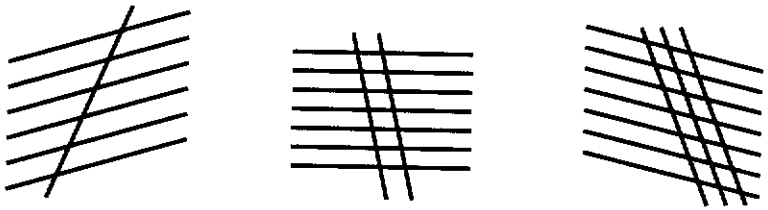


Fig. 2

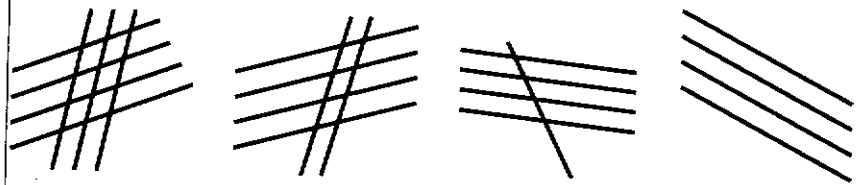


Fig. 3

Fig. 4

A3 – Discovering Number Properties - I

1. Prologue

In this era of increasing awareness for the rights of the child, it will not be inappropriate to highlight the lack of opportunities for gifted children, particularly in mathematics, in the present-day schooling situation, therefore, some remedial measures are suggested.

Gifted pupils in mathematics in India are yet to be given as much extra direction as in advanced countries. For instance, in the erstwhile USSR, mathematical talent is valued as highly as other valuable resources. There, special schools, named after Kolmogorov, a renowned Russian mathematician, are run by the State exclusively for the gifted pupils in the field of mathematics.

In developing countries, suffering from a high rate of illiteracy, the gifted children rarely find themselves in a milieu of acceptance and appreciation with adequate opportunities to exercise and exhibit their mathematical talent as they form a small minority. The reason is that the needs of the majority are so pressing and the resources at the disposal of the schools so limited. Moreover, the school and social environment is so non-mathematically oriented that it does not shock people to see that the mathematically gifted are most often left to fend for themselves. This results mostly in early discouragement of the precociousness in children and their consequent diversion to other fields which evoke greater expectancy and provide greater encouragement. Neither getting high marks is a sure sign of giftedness or awarding high marks a sure means of recognition and care of the gifted.

This situation will have to change. And that is possible only if teachers are trained to know how to detect and foster mathematical talent in children and realise that detection and care of the gifted should start right from their kindergarten and primary years.

In the words of Iburu Masaka, the economic wizard of Japan 'kindergarten is too late'. Mathematical talent is like musical and artistic talent which need nurturing from one's early years of childhood.

I would like to share with people, parents and teachers in particular, a programme which I launched successfully in Nigeria in pursuance of the great objective of helping children discover their mathematical talent and exhibit them to others. *Though discovery-oriented methods have of late become the fashionable talk of the day in the educational world, one rarely finds them in practice in the classroom.* For most of the teachers continue to feel satisfied with telling and helping children 'learn' by imitation and repetition of what they are told and what appears in print. This of course, is widely believed to be the easy and sure way to secure pass marks in the examination.

It is no wonder that some articulate teachers take pride in asking bluntly, 'What is there for children to discover?' They think that it is their duty to raise the question, 'Why should the children be allowed to waste time in the process of discovering ?' What an attitude it is which the gifted children face especially in this age when discovery and invention alone ensure survival.

The teachers, however, are not to be blamed as they have had, by and large, no such experience either in their school days or in their training periods, pre-service or in-service, and they naturally fail to understand the importance of discovery-oriented learning.

2. What is discovery?

What is meant by discovery is that children are to be guided to find out concepts by themselves without being tutored. The fact is that they do find out a lot, given a stimulating environment and suitable opportunities. Not only that, they acquire skills faster, because they are better motivated.

Mathematics like science is based upon experience particularly during the early years of schooling and as such that phase abounds in opportunities for discovery by gifted children right from their pre-secondary years. Such

opportunities ought to be created and offered to the children in order to promote their self-confidence and hence their mental growth.

An attempt is made herein to show how, just with the skills of addition and subtraction, multiplication and division, children enjoy discovering numerous number properties or relations.

To start with, the children should be able to identify the basic kinds of numbers they often come across in mathematics first by means of concrete experiences and then in terms of their number names or numerals, tacked on as the latter are to a base.

3. Strategies

(i) *Even or Odd:* Allow children to use *bottle tops* or *coloured plastic cubes* of the same size for setting up arrangements. Activities for concretisation through arrangements are pictured in Fig. 1 for identification of kinds of number. If the objects in a collection are paired off perfectly the number of the collection is *even.*

One pair gives the 1st even number 2 (1 × 2). Two pairs give the 2nd even number 4 (2 × 2). Three pairs give the third even number 6 (2 × 3) and so on. With this pattern, the even numbers in any position or rank can be given without listing them in a sequential order.

If the objects in a collection need one more object to complete the pairing process, the number of the collection is *odd.* One pair less one gives the first odd number 1 or 1+0 = 1. Two pairs less one give the second odd number or 2+ 1 = 3. Three pairs less one give the third odd number of 3+2=5 and so on. With this pattern, the odd number in any position or rank can be given without listing the odd numbers in order (see Fig. 2).

If the objects in a collection can be arranged in a *rectangular array* (having more than one row and equal number of objects in each row and no row generally having less than 2 objects), the number of the collection is said to be *rectangular* or *composite. The composite number do not form an alternating pattern as in the case*

of even and odd numbers. However, they can be listed in order (see Fig. 3).

Incidentally, children recognise *factors* or *divisors* and *multiples of a number.* Taking Fig. 4 they learn to say that six is a multiple of 2 or 3 and 2 and 3 are factors of six. They also discover that *some composite numbers (e.g.* 24) *can be displayed in different rectangular arrays* (e.g. 12 × 2, 8 × 3, 6 × 4) revealing more factors.

(ii) *Prime Numbers: If the objects in a collection can be arranged only in a single row and not in an array of more than one row, the number of the collection is non-rectangular* or *prime.* Incidentally children see that though any collection can be arranged in a row, any *collection whose number is prime can be arranged only in a unique manner in a single row.* They also realise that *a prime number has only two factors* 1 *(one) and the number itself, leading to the understanding that a number having only two factors namely* 1 *and the number itself is prime and a number having more than two factors is composite.* They also decide that 1 *is neither prime nor composite as it has only one factor viz* 1. The prime numbers too do not form a pattern, though they can be listed in order (see Fig. 5).

(iii) *Square Numbers: If the objects in a collection can be arranged in a square array (having as many rows as there are objects in a row), the number of objects in the collection is a square number* (see Fig. 6).

Two rows of two objects each give the second square number 4 (2×2 or 2^2)

Three rows of three objects each give the third square number 9 (3×3 or 3^3).

Four rows of four objects each give the fourth square 16 (4×4 or 4^4) and so on. Note that since $1 \times 1 = 1$, 1 becomes the 1st square number.

(iv) *Cube Numbers: If the objects in a collection can be arranged to form a block having as many layers of square array as is the number of objects in each of its arrays, then the number of objects in the collection is a cube number* (see Fig. 7).

Two layers of two rows with two objects in each row give the **2nd** cube number 8 ($2\times2\times2$ or 2^3); three layers of three rows with three objects in each row give the third cube number 27 ($3\times3\times3$ or 3^3) and so on. Note that since $1\times1\times1 = 1$, 1 becomes the 1st cube number.

(v) *Triangular Numbers: If a collection can be displayed in a sequence of* ***rows*** *of objects in such a way that their numbers in the successive rows are in the order of natural numbers* 1,2,3 *etc., the arrangement takes the shape of* a *triangle* and *hence the number of objects in any such collection* is *called a triangular number* (see Fig. 8).

If two collections having the same triangular number are combined they can always be arranged to form a special kind of rectangular array where the number of (horizontal) rows is one less than the number of objects in a row, as seen in each array of the second *line-up* of the Fig. 9. Each array in the second line-up gives an *oblong number. The pattern of the oblong numbers making listing of triangular numbers easy.*

The first triangular number is 1/2 of the 1st oblong number 1×2; the second triangular number is 1/2 of the 2nd oblong number 2×3; the third triangular number is 1/2 of the 3rd oblong number 3×4 and so on. The triangular number in any position or rank can easily be given.

4. Faster Learning

As activities suited to the maturity levels of primary school children and within easy reach of every primary school teacher and every parent at home, precede identification and naming of the basic kinds of numbers, learning is faster, delightful and permanent.

Children can be seen reeling off without diffidence or hesitation the list of at least first ten numbers of each kind. Given the position of any particular kind of number, they can name it without listing, *except in the cases of prime and composite numbers.*

Discovering Number Properties–I

Even Numbers

Fig. 1

Odd Numbers

Fig. 2

Composite Numbers

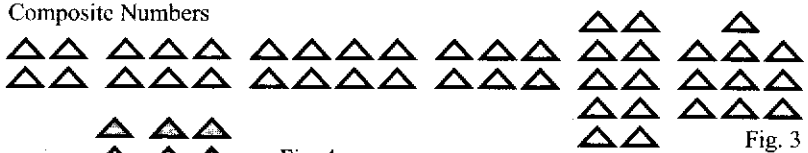


Fig. 3

Fig. 4

Prime Numbers

Fig. 5

Square Numbers

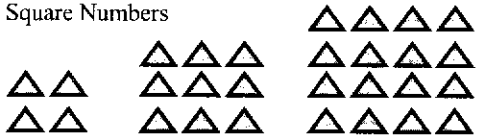


Fig. 6

Cube Numbers

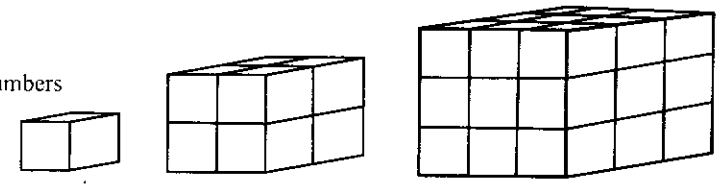


Fig. 7

Triangular Numbers

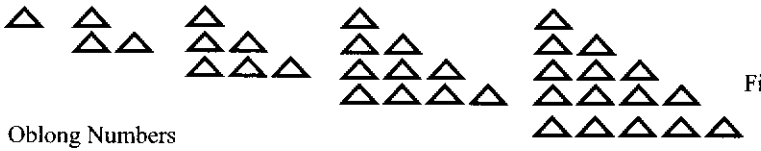


Fig. 8

Oblong Numbers

Fig. 9

A4 – Discovering Number Properties - II

1. Prologue

Once children learn to identify different kinds of numbers, they find themselves poised for making exciting discoveries of relations or properties of numbers, using only their skills of performing four operations. Whenever more than two numbers are involved, addition and multiplication will mostly be enough.

2. Number Discovery Cards

Number discovery cards (NDCs) are prepared and given to them. Each card shows what numbers they have handled to discover the properties or relations. *The cards have the characteristic of carrying no verbal explanations or questions.*

The discovery card-1 directs the attention of the card to all the pairs of consecutive natural numbers. I shall state herein some of the discoveries made by a few bright children of primary V in Nigeria almost in their own language, on the occasion of the celebration of the IYC in May 1979 in my training college there with the cooperation of primary schools around. Sometimes a few eliciting questions needed to be put to help children communicate.

1. Working: $1 + 2 = 3$; $2 + 3 = 5$; $4 + 5 = 9$; $5 + 6 = 11$;....
   Discovery: Add a number of its next number. We get an odd number.
2. Working: $1 \times 2 = 2$; $2 \times 3 = 6$; $3 \times 4 = 12$; $4 \times 5 = 20$;....
   Discovery: Multiply a number by its next number. We get an even number.
3. Working: $2 - 1 = 1$; $3 - 2 = 1$; $4 - 3 = 1$; $5 - 4 = 1$;....
   Discovery: Take a number away from its next higher number. We get always 1.

4. Working:

1)2(2	2)3(1	3)4(1	4)5(1
2	2	3	4
0	1	1	1

Discovery: Take a number and its next. Divide the bigger by the smaller. The quotient is always 1 and the remainder is always 1. This is not true for 2 and 1.

Once motivated the children have special flair for discovering patterns and that makes them succeed quite remarkably. Eager parents and teachers have the thrill of seeing *what our primary school children are capable of and thereby get a better understanding of their untutored abilities in mathematics*

The other discovery cards NDCs 3 to 5 which I used in the IYC programme are given below for use by children. *When more than two numbers are circled, let them know that they can use all of them or groups of them in discovering number relations or properties and adding a number to itself or multiplying a number by itself are also allowed.*

Similar discovery cards can also be prepared for other areas of mathematics. Elementary number theory is the most popular and of interest to everybody and hence it has been given preference. Some of the bright students can be seen raising more questions not confined to the specific directions as indicated in the discovery cards and coming out with breath-taking findings. They also enjoy displaying their discoveries through arrangements of objects, wherever possible.

It is the birthright of every mathematically gifted child to be given opportunity to develop his research potential to become junior mathematicians before blossoming into senior mathematicians later. Provision of a mathematics club with a small library stocked with enrichment books is the least the gifted have a right to expect in a school, if not at home.

3. TV Discovery

Programmes such as these can better be shown live on TV, once a fortnight with children seen vying with each other in their discoveries and they would certainly electrify the atmosphere in homes inducing emulation in many other children.

* Later developed into an enrichment book for children in dialogue mode under the caption *Romping in Numberland* written and published by the author in 1988.

Discovering Number Properties–II

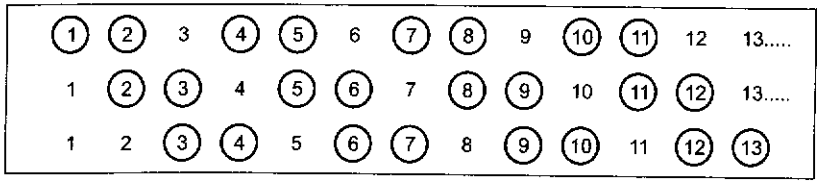


Fig. 1

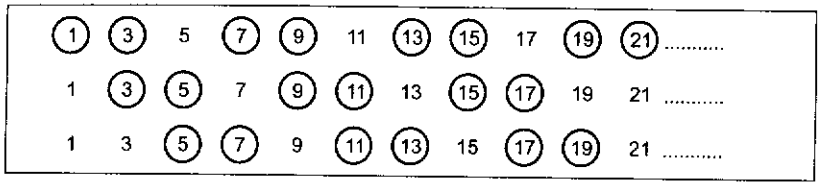


Fig. 2

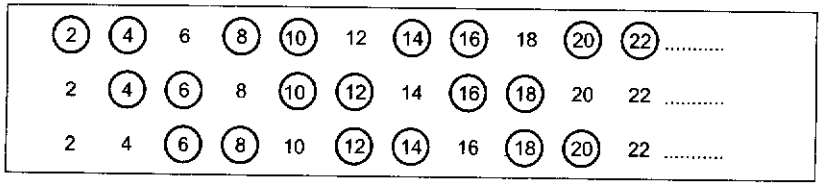


Fig. 3

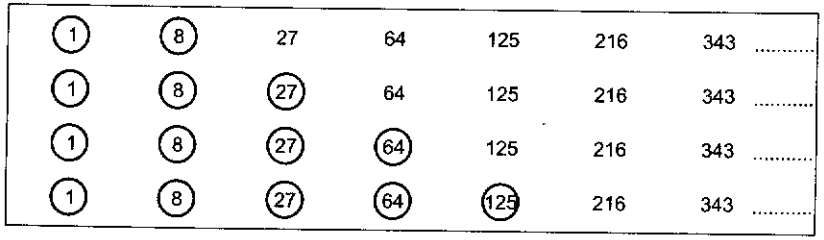


Fig. 4

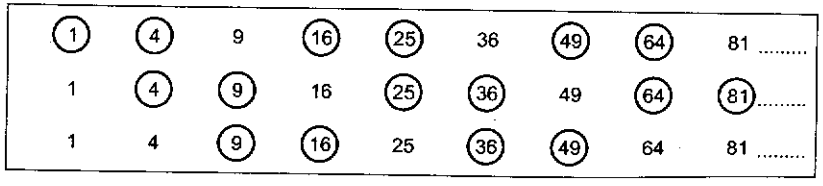


Fig. 5

A5 – Games to Learn Integers

1. Prologue

There is a dramatic way of making children realise the need to associate direction with numbers in certain situations. This can be done at home or school.

Ask a child whether he can act according to a simple command. He would say, 'Surely'. If a few other children are there to witness the 'drama', it would be quite welcome. Tell the child, 'Move two steps.'

If he is not alive to the implication by virtue of having found himself earlier in a similar situation, he can be seen to move guilelessly two steps forward. Make the observation that he has done something which he has not been told to do. One or two of the children witnessing the 'drama' cab be seen to vouch for your observation by commenting that you have only asked him to move *two steps* and *not two steps forward.*

The performing child realises that he has simply assumed something unwarily and he cannot act without the specific and explicit mention of the direction associated with the number of steps. Children get introduced to the idea that mere numbers are not enough in some situations which require the *association of direction with numbers.*

The world of opposite such as *forward* and *backward,* right and left, gain and loss, above and below, after and before, etc. surfaces in the minds of children. The great mathematician Bhaskara of A.D. 12th century called them *dhana rashi (asset numbers)* and *him rashi (liability numbers).*

They came to be accepted and called, about three centuries later, by European mathematicians as positive or plus numbers and negative or minus numbers respectively.

Once this general idea is understood through situations with opposites, an interesting game can be set up. *Playing a game is to some extent akin to axiomatisation in mathematics. The most desirable feature in a game approach is the acceptability of rule or rules with little resistance and the study of the consequences through ready involvement.*

Take objects like bottle tops wherein two sides, one side and its opposite, can be distinguished easily or cardboard bits with opposite sides *coloured differently*. One side is taken as *positive* whereas the other (or the opposite) side is taken as *negative*.

Take some of the objects to start with. Situations arise, when set-ups are made with all of them showing the same side (positive or negative) or with some of them showing one side and the rest the other side.

2. The games

The game is about giving the *net value of* a set-up after accepting the rule that *two objects one showing one side and the other the opposite side, cancel each other in value. In other words, the inclusion or exclusion of such pairs does not alter the value of a set up.*

Let us consider some typical situations. Assume that cardboard bits are used, each with one side lettered 'P' (positive) and its other side 'N' (negative). The situations with the net values of set-ups are illustrated along with the corresponding verbal statements and their symbolic translations.

Positive three together with *positive two* gives *positive five.*

$+3 + 2 = 5$ or $(+3) + (+2) = 5$

Negative three together with *negative two* gives *negative five.*

$-3 - 2 = -5$ or $(-3) + (-2) = -5$

(Excluding as many pairs 'P' 'N' as possible), *positive three together with* negative *two gives positive one.*

$+3 -2 = -1$ or $(+3) + (-2) = +1$.

Negative three with *positive two* give negative one.

$-3 + 2 = 1$ or $(-3) + (+2) = -1$

Negative two together with *positive two* is zero

$-2 + 2 = 0$ or $(-2) + (+2) = 0$

The exciting part of the game surfaces when *subtraction is considered through 'taking-away situation'.*

The removal of *positives* from *positives* giving positives or *negatives* from *negatives* giving negatives is simple.

Taking away positive two from *positive three* gives *positive one.*

$(+3) - (+2) = +1$.

Taking away negative two from *negative three* gives *negative one.*

$(-3) - (-2) = -1$.

Can we remove negative one from positive two? This challenging situation makes interesting use of the rule. As *there is no negative, a negative is needed to effect the removal. But a negative cannot be brought into the set-up without its positive counterpart, as only when such a pair is so brought in, the value of the set-up will not get changed.*

3. Another situation

Children pick up soon verbalisation and symbolic translation and *discover the 'change the sign and add' rule, as a short-cut,* In the set-ups involving subtraction, instead of bringing in suitable number of pairs to remove the required number of positives or negatives, children pick up the required number of positives or negatives to be removed, turn them over, include them in the set-up and find the net value result to be the same.

Children get excited and are seen raising questions such as 'How to remove positive 3 from negative 2 ? How to remove negative 3 from positive 2 ? How to remove positive 2 from zero? How to remove negative 2 from zero?' and so on. *They are immensely pleased to find the 'bringing in the required pair' approach as well as 'the short-cut' working.*

Soon the children develop readiness to answer questions put in verbalised and then symbolised form without the use of objects but with or without visualisation. This transition from the concrete to the abstract is the hall-mark of mathematics education.

4. Directed numbers in multiplication

Multiplicatcn of directed numbers can be presented as repeated addition with the required modification.

When you ask a child to pick up a certain number of positives or negatives and place them in the required number of times, the child can be seen or helped to ask if the objects should be placed on the *same side showing up while picking should be changed to the opposite side.*

Placing 3 *positives on the same side twice, the net value is positive* 6.

(+3) (+2) = + 6

Placing 3 *positives on the opposite side twice, the net value is negative* 6.

(+3) (–2) = 6.

This net value is also got from placing 3 *negatives on the same side twice.*

(–3) (+2) = – 6.

Placing 3 *negatives on the opposite side twice, the net value is positive* 6.

(–3) (–2) = + 6

Division situations can similarly be devised and structured.

5. Some comments

Teaching directed numbers have always created problems and the game approach is only one of the approaches in presenting directed numbers and their operations. Compared to other approaches like 'combined changes study', 'number line', 'extended pattern' etc. the appeal of the game approach is great and instant.

A majority of students settle down fast to grasp the principles, as it involves *low learner-resistance.* Even children in lower primary schools, find learning of directed numbers great fun, as they by themselves discover all the operational rules just by accepting one rule of the game and familiarising themselves with the language, verbal as well as symbolic.

It is of course worthwhile to present other approaches and concrete situations as well so that the topic emerges in its full splendour of abstraction.

Games to Learn Integers

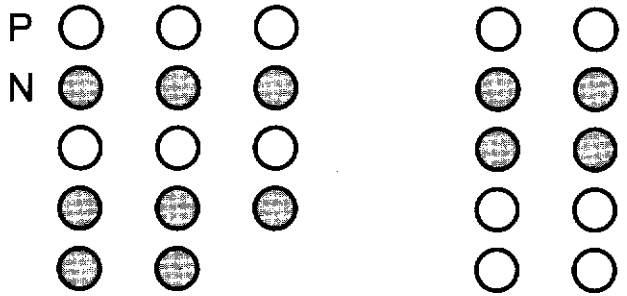


Fig. 1

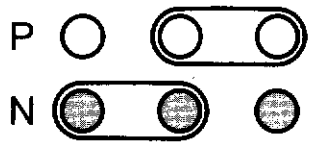


Fig. 2

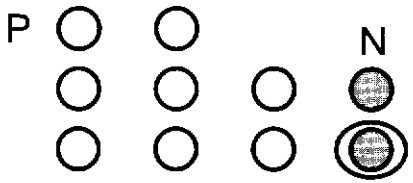


Fig. 3

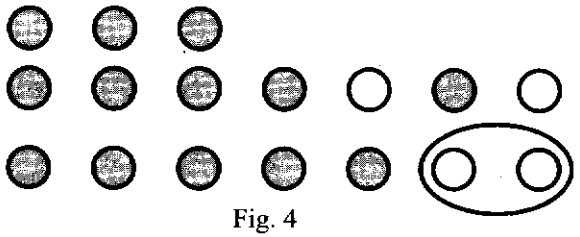


Fig. 4

A6 – Surprises with Clock Arithmetic

1. Prologue

Some familiar objects and their functioning have rich structural relations below their surface appearances which can be pressed into service to provoke mathematical thinking. One such object is the clock with its set of 12 numbers on its face. With the proliferation of digital clocks and watches the analogue ones showing time by movement of hands may in the near future be relegated to the limbo of antiques. Mathematics of clock face numbers if appreciated will prevent this disaster. A model clock face in cardboard with movable hands used in primary schools would be of help in getting the required experience. There is no need for the minute hand in this arithmetic and hence it can be removed.

Twelve hour clocks are more common than 24 hours clocks. So give the child a 12-hour clock and let the reckoning of time be done in hours only. Children should first be helped to recall *the difference between an instant and an interval.* Two hours may mean two hours duration (interval) or two o'clock time instant.

2. Clock arithmetic

If at a certain time, the clock shows 8 o'clock then five hours hence it would be 1 o'clock . This can be written 8+5=1 provided *the '+' sign here is taken to denote not ordinary addition but clock addition.* For the distinction we may write (+) for clock addition and (×) for clock multiplication. All other clock additions can be obtained otherwise. Children can be asked to find all the clock addition facts and present them in the form of a composite table, called *Cayley table* (see Fig. 1).

Having done clock addition, a natural question about clock multiplication would arise. What does multiplication mean in the set or system of whole number 0, 1, 2, 3, 4 etc.? It is simply repeated addition of a whole number, *with the number of repetitions, the whole number and the sum becoming*

respectively the multiplier; the multiplicant and the product. Starting say from 9 o'clock, nine hours hence, it will be 6 o'clock again after 9 hours it will be 3 o'clock, yet again after 9 hours it will be 12 o'clock and so on. These can be written thus

$9 + 9 = 6$ or $9 \times 2 = 6$
$9 + 9 + 9 = 3$ or $9 \times 3 = 3$
$9 + 9 + 9 + 9 = 12$ or $9 \times 4 = 12$

and so on, the '+' and '×' signs here representing clock addition and clock multiplication respectively. Children can be allowed to build the clock multiplication tables and display them in composite form as before (Fig. 2) with a word of caution to exercise greater care in fixing products.

These tables provide a rich fare for mathematical observation by *comparing the behaviours of the set of whole numbers and the set of clock numbers.* The sum of any two or any number of whole numbers, for that matter is a whole number. Similar is the case with clock face numbers under clock addition. Under multiplication also, both the sets have the same behaviour.

Consider the pattern in the sums of two clock numbers, one of which is 12, 12 + 1 = 1: 12 + 2 = 2 etc. Are these not like 0 + 1=1; 0 + 2 = 2, etc. So *clock number* 12 *behaves like the whole number 0.*

The pattern seen in $1 \times 12 = 12$, $2 \times 12 = 12$, $3 \times 12 = 12$, etc. or in $12 \times 1 = 12$, $12 \times 2 = 12$, $12 \times 3 = 12$ etc., also confirms that 12 behaves like 0. So *it is appropriate to recognise or identify clock numbers* 12 *as the zero of the system of clock face numbers.*

As in the system of whole numbers, there is no ambiguity about the sum of any two clock numbers.

3. Order in Clock Numbers

Subtraction exposes interesting situations; in the set of whole numbers, any whole number cannot be subtracted from any other whole number. We have to identify the greater and the smaller and subtract the smaller from the greater. To remove the restriction, the set of integers 0, + 1, + 2, + 3 etc. (+ read

as plus or minus), that is, positive or negative whole numbers were invented. In this extended system, any integer can be subtracted from any other integer.

Is there such a restriction about clock subtraction ? Allow children to find it out.

Now in clock subtraction; 5 –1 = 4, also 1–5 = 8 (5 hours before 1 o'clock is 8 o'clock) 10–3 = 7, also 3–10 = 5 and so on. This is a new mathematical experience for those who know only whole numbers and subtraction with them. *The system of clock face numbers requires no extension as has been shown above.*

The integers have order among them like the whole numbers. That is to say that, if any two integers are considered, one of them will be greater than the other, in case they are not equal. Does order prevail in the system of clock face numbers? Can we say, for example, 8>4 (8 is greater than 4)? If 8>4, then 8+8>4+4 should hold good. But 8+8=4 and 4+4=8, hence 8+8>4+4 gives us 4>8 showing conclusively *that there is no order in the system of clock face numbers.* In other words, *clock face numbers cannot be arranged in ascending or descending order.* What a surprise!

Another shake-up experience awaits children when they attempt to find what happens when one clock face number is divided by another clock face number. Where a whole number can be divided (without remainder) by another whole number, *the quotient is obtained uniquely.* Ask children to find if such a behaviour obtains in the system of clock face numbers, by studying the clock multiplication table for clock division.

8 – 2 = 4 or 10 (4 × 2 = 8, 10 × 2 = 8)
6 – 3 = 2,6 or 10 (why)
8 – 4 = 2,5,8 or 11 and so on

Children discover that *quotients in the system of clock face numbers are not* unique.

4. Whole numbers vs. clock face numbers

The system of clock face numbers has another very surprising property not possessed by whole numbers. *When two whole numbers other than zero are multiplied, the product is always*

a non-zero whole number. So we are entitled to say that *if the product of two whole numbers is zero, at least one of them should be zero.* Can a similar statement be made in the case of clock face numbers? Consideration of examples such as 6 × 4 = 12, 8 × 3 = 12 etc. shows that such a statement cannot be made with regard to clock face numbers. In other words in *the clock face number system zero may have all its factors non-zero unlike in the whole number system.*

Mathematical growth of a bright primary school child would be incomplete, if this experience with clock face numbers does not form part of his or her exit behaviour. *It would also inculcate in the child an attitude of expectancy for surprises in number systems and wariness with assumptions. The child will realise that the word number is an umbrella term, rich with numerous associations.*

Children can be set the interesting project of building 'clock arithmetic' with odd number of 'clock numbers 1 to 7, 1 to 5, etc. and with even numbers of 'clock numbers 1 to 6, 1 to 10, etc. and children can be seen to discover that when the count of 'clock numbers considered is prime *the system of odd number of clock numbers behaves like the system of whole numbers in respect to addition and subtraction, multiplication and division.*

When the last clock number is taken as zero itself, then the clock arithmetic get recognized as *modular (or measuring out) arithmetic of remainders (or residues to use mathematicians' term)* developed by the great mathematician Gauss and finding an important place in Number Theory. Another great surprise awaits the child when it sees that each *number in modular arithmetic is not just an individual number but one naming a class of numbers as, for instance, 1 of clock arithmetic names the class of numbers relating to timings 1 hr., 13 hr., 25 hr., 37 hr., 49 hr., etc.* (as at these timings the clock will show 1 o'clock).

Surprise with Clock Arithmetic

+	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	1
2	3	4	5	6	7	8	9	10	11	12	1	2
3	4	5	6	7	8	9	10	11	12	1	2	3
4	5	6	7	8	9	10	11	12	1	2	3	4
5	6	7	8	9	10	11	12	1	2	3	4	5
6	7	8	9	10	11	12	1	2	3	4	5	6
7	8	9	10	11	12	1	2	3	4	5	6	7
8	9	10	11	12	1	2	3	4	5	6	7	8
9	10	11	12	1	2	3	4	5	6	7	8	9
10	11	12	1	2	3	4	5	6	7	8	9	10
11	12	1	2	3	4	5	6	7	8	9	10	11
12	1	2	3	4	5	6	7	8	9	10	11	12

Fig. 1

x	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	2	4	6	8	10	12
3	3	6	9	12	3	6	9	12	3	6	9	12
4	4	8	12	4	8	12	4	8	12	4	8	12
5	5	10	3	8	1	6	11	4	9	2	7	12
6	6	12	6	12	6	12	6	12	6	12	6	12
7	7	2	9	4	11	6	1	8	3	10	5	12
8	8	4	12	8	4	12	8	4	12	8	4	12
9	9	6	3	12	9	6	3	12	9	6	3	12
10	10	8	6	4	2	12	10	8	6	4	2	12
11	11	10	9	8	7	6	5	4	3	2	1	12
12	12	12	12	12	12	12	12	12	12	12	12	12

Fig. 2

A7 – New Wine in Old Bottles

1. Prologue

It is fashionable with politicians and businessmen to put old wine in new bottles, but with mathematicians it is the other way round. Mathematicians are fond of putting new wine in old bottles for purposes of consolidating the gains of the past and opening the possibilities of making new advances. Initially it may be trying for the beginners but soon it turns out to be a lesson in appreciation of economy and power of mathematical thinking.

Children's first exposure to use of letters in mathematics occurs in generalized arithmetic and solution of equations. *Letters are introduced to serve as unknowns or variables representing numbers. But use of letters to represent non-number situations like switches in circuit, sets and statements* has not yet become common place in school curriculum. So children naturally feel a little uncomfortable till they settle down to the revolution in the use of letters and signs. What strikes one as very odd above all is the non-number use of numerals 1 and 0 to represent *bi-state* situations such as *yes* and *no, true* and *false, belongs to* or *does not belong to* etc. What appears at first to be the misuse turns out to be an instance of magnificent use.

2. Algebra of switches

In every home today where transistor, radios and torches have become common place articles of use, children have ready access to electric cells and find joy in experimenting with them. With a few pieces of insulated copper wire, a torch bulb with a holder and a few simple or improvised switches, children can be provided with a very valuable opportunity to experience *mathematisation of circuits*. See figs. 1 and 2.

When more than one switch are used, the connections, as children know from their school science lessons, can be only of two basic types; parallel and series and their combinations.

Help children set up (1) a parallel connection with two switches, and (2) a series connection with two switches. Their diagrammatic representation are given on page 37. See figures 3 and 4.

a and *b* in figures 3 and 4 represent two switches in each circuit. Children can be expected to know or enabled to observe by operating switches that in *a parallel circuit, the current does not flow and the bulb is not lighted only when both the switches are off. But in a series circuit, both the switches should be on for the current to flow and light the bulb.*

Representing *on* state (or flow of current situation) by 1 and *off* state by 0, the various states of switches in each kind of basic connections with the corresponding outcomes can be mathematised in the form of tables as shown below. Ask children to complete the tables with or without actual operations of switches. Children should be getting the following tables:

Table 1

Switches in parallel		
a	b	outcome
0	0	0
0	1	1
1	0	1
1	1	1

0-off, 1-on

Table 2

Switches in Series		
a	b	outcome
0	0	0
0	1	0
1	0	0
1	1	1

0-off, 1-on

The new use of 1 and 0 is like putting new wine in old bottles. Through the table of outcomes for parallel connection, children can be seen recalling the addition table with 0 and 1 except for the last item involving 1 and 1 ; for $0 + 0 = 0$; $0 + 1 = 1$; $1 + 0 = 1$. The table of outcomes for series connection recalls the multiplication table with 0 and 1; for $0 \times 0 = 0$; $0 \times 1 = 0$; $1 \times 0 = 0$; and $1 \times 1 = 1$. So what is the new wine in the old bottle? *+ sign can in this context be allowed to represent parallel connection (or combination) of two switches and X sign series connection (or combination) of two switches.* The tables 1 and 2 can now be rewritten as follows:

Table 1

a	b	a+b
0	0	0
0	1	1
1	0	1
1	1	1

Table 2

a	b	ab (i.e., a.b or a × b)
0	0	0
0	1	0
1	0	0
1	1	1

To ensure familiarisation of new use of letters, numerals, 1 and 0 and signs + and × (or .) and the consequent new outcomes, help children to set up switching models or draw switching diagrams for the following situations $a + a$, aa, $a' + a$, aa' , $1 + a$, $1 .a$, $0 + a$, 0, a, $(a')'$ and evaluate or simplify them. It should be explained to the children that (1) repetition of letter means that more than one switch are on or off at the same time, (2) a switch that is *always on* is represented by 1 and a switch that is *always off* by 0 and (3) a and a′ *represent switches which are not on or off at the same time*, that is to put in another way, when a is on, a' is off and when a is off, a' is on. By this time children should have realised that except 1 and 0, letters can take no other numerals for values as they would be meaningless in these bi-state situations.

3. Algebra of sets

After experience with circuits, encourage children to consider a *set of objects* and all the sets including the *empty set* that can be formed by taking 0,1,2 etc. of the objects of the whole sets in all possible ways. Ask children to represent the sets by letters *a, b,* etc. If certain objects of the whole set form a set, say a, all the remaining objects of the whole set would form another set denoted by a' which is *complementary* to the set a. Give new roles for 1 and 0 to represent respectively that an object "belongs to" and that an object "does not belong" to a set. Ask children to complete the tables to show the outcome of union ($\cup$) and intersection ($\cap$) of sets, after examining to find out if the children realise that (1) there is only one case when an object will not be in the union set, and that is when the object does not belong to either a or b and (2) that there is only one case when an object belongs to the

intersection set and that is when the object belongs to both the sets. See figures 5 and 6.

In figures 5 and 6, points in the circular region are considered to represent objects of sets and points in the rectangular regions objects of the whole set; a and b are two sets of objects. The shading in Fig. 5 shows the set of objects got by union of sets a and b and the shading in Fig. 6 the set of objects got by intersection of sets a and b.

Table 3

a	b	a ∪ b
0	0	0
0	1	1
1	0	1
1	1	1

Table 4

a	b	a ∩ b
0	0	0
0	1	0
1	0	0
1	1	1

Children cannot avoid comparing the tables 3 and 4 respectively with tables 1 and 2 and finding them *surprisingly alike* except for the change in context. So there is again a chance to put new wine in old bottle. + and × can now be assigned yet other new roles of union of sets and intersection of sets respectively. Since $a.S = a$, $b.S = b$, $a+S = S$, $b+S = S$ etc. 1 *can be interpreted to represent the whole set* (or the universal set to use mathematicians' language). Children might be knowing that a set can be without objects. Such a set is called the empty set and is denoted by 0. Since $a + 0 = a$, $b + 0 = b$, $a.0 = 0$, $b.0 = 0$ etc. *0 can be interpreted to represent the empty set.* As before let children draw diagrams, interpret and evaluate or simply $a+a$, aa, $a+a'$, aa', $1+a'$, $1.a$, $0+a$, $0.a$ and $(a')'$.

4. Algebra of Statements

Finally, show children yet another use of letters to represent *statements*, which can be either true or false, such as

a : he passes in English

b : he passes in Mathematics

Let 1 and 0 represent in this context true and false respectively. Any two statements can be joined by OR (used

in the sense *in this or that or both)* and by AND. In the case of a compound statement formed by connective OR, help children to realise that there is *only one case when the compound statement is false and that is when the statements are individually false and in the case of compound statement involving the connective AND, there is only one case when the compound statement is true and that is when both the statements are individually true.* Let children complete the tables. They should be getting the following:

Table 5

a	b	a OR b
0	0	0
0	1	1
1	0	1
1	1	1

Table 6

a	b	a AND b
0	0	0
0	1	0
1	0	0
1	1	1

Once again the tables 5 and 6 are exactly like the tables 1 and 2 except for the change in context. So children get yet another chance to put new wine in old bottle. + and × in this context stand respectively for the connectives OR and AND in logic. Now what does a' represent? a' means *negation* of a given statement a. If a stands for the statement 'he speaks French' a' would stand for the statement, 'he does not speak French'. Children then easily see that $a+a'$ is *always true* and so can be represented *by 1* and aa' is *always false* and so can be represented *by 0*. As before, let children interpret, evaluate or simply the following compound statements $a + a$, aa, $a + a'$, aa', $1 + a$, $1.a$, $0 + a$, $0.a$, $(a')'$. They would have found that $a + a = a$, $aa = a$, $a + a' = 1$, $aa' = 0$, $1 + a = 1$, $1.a = a$, $0 + a = a$, $0.a = 0$, in all the three contexts. A comparison of these laws with those of ordinary algebra is of great educational value.

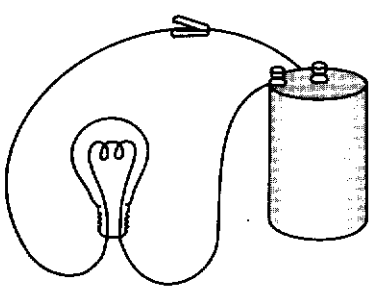
5. Abstract thinking

This valuable experience of using letters and signs in different contexts and discovering that *different contexts have the same combination behaviour or structure as mathematicians would call it instills readiness in children to appreciate the need for*

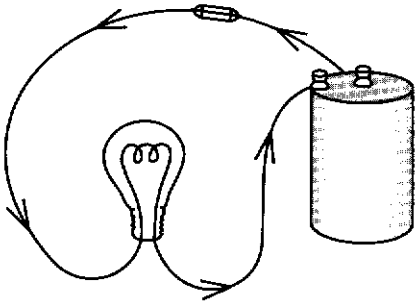
axiomatisation for presenting abstract mathematics. Taking letters to represent indeterminate things and giving rules (or axioms) for manipulations (or operations) with them leads to an exercise in abstract mathematics. *Without different contexts having a common structure, the significance of context-free development of ideas cannot be easily realised.*

Accepting entities recognizable only by their operations and relations inculcates easily in one the feel for abstraction and appreciation of its beauty and power. By considering *a, b,* c etc. as indeterminate elements with operations + and × defined as shown in the above described models, we get the famous algebra named after the 19th century British mathematician, *George Boole* who was the first to show how an algebra can be developed with letters representing non-numbers. Through Boolean algebra it is easy to bring home to school children the role of abstract thinking in mathematics. Each of the above contexts or models becomes a concrete realisation of this abstract algebra and *problems in one context can be solved in another context.* Moreover, the inclusion of computer science in school curriculum necessitates early exposure of children to Boolean algebra which surpasses ordinary algebra in simplicity and symmetry of operations.

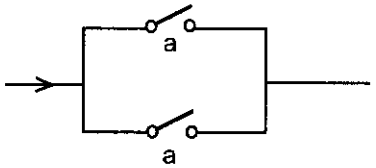
New wine in Old Bottle



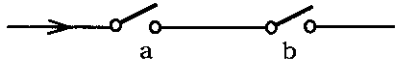
OFF POSITION OF THE SWITCH
Fig. 1
There is no flow of current through the circuit and the bulb is not lighted



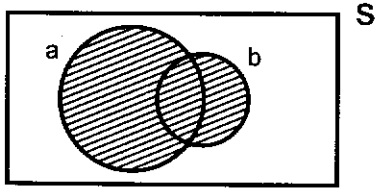
ON POSITION OF THE SWITCH
Fig. 2
There is flow of current through the circuit and the bulb is lighted



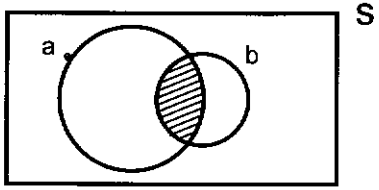
PARALLEL
Fig. 3



SERIES
Fig. 4



A ∪ B
Fig. 5



A ∩ B
Fig. 6

A8 – An Integrated View of Mathematics

1. Prologue

As children climb up the ladder of mathematical learning, they face cognitive problems in revising their notions about mathematics. In primary years, they get the picture of mathematics as a subject of numerical computations. In middle years, they have to reconcile themselves to algebraic expressions and equations. In high school years, they have to meet the challenge of proving in mathematics. The transition is too sudden and the teachers' motivating skills are so inadequate that many children fail to respond to the new challenges in the study of mathematics. One way of meeting this educational problem in mathematics is to intersperse learning with examples, counter examples, conjectures, proving by exhaustion in finite sets, local axiomatics etc. at all levels.

2. A truth about any set of eight persons

Children love company and can reel off the names of their friends. Ask a child to list names of his/her eight friends. You can surprise the child by *claiming* that you can make a truthful statement about them irrespective of the fact that they are strangers to you. The statement is that *at least two of his/her eight friends are born on the same day.* If the child is able to, he/she can verify the statement by actual collection of dates of birth of his/her friends. Now tell the child that your statement is true about *any set* of eight persons, known and unknown, living or dead, or to be born for that matter. The child will naturally feel baffled and wonder how it can be applicable to every set of eight persons.

Allow the child to find out the reason. If need be, throw the hint: 'As an extraordinary case, assume that each child is born on a different day of the week'.

The child can be seen to find his/her way out. Seven days of the week will be associated with seven persons and so the eighth person should be born on anyone of the days of the week, thereby proving the statement. You can observe an intelligent child making a similar statement for thirteen persons, viz., that at least two of thirteen persons are born in the same month. And so on. The reasoning is based on acceptance of some initial statements: 'There are seven days in a week', 'There are twelve months in a year' etc. Such initial statements are called *axioms, postulates* and *assumptions* in mathematics which are to be accepted without argument. This experience dramatises the need for reasoning or proof in mathematics.

3. Local axiomatics

Let us consider another situation. Children easily learn even in the primary school that the sum of two odd numbers is even, the sum of an odd number and an even number is an odd number and so on. It will help children to proceed from these statements to prove that the sum of any three odd numbers is odd. The sum of three odd numbers involves the sum of two odd numbers and a third odd number. Since the sum of two odd numbers is even, it reduces to summing an even number and an odd number and this gives an odd number. And thus the statement that the sum of three odd numbers is an odd number stands proved. This is exposure to *local axiomatics.*

Counter example

One more instance can be cited, children learn about prime numbers and composite numbers. When asked if the sum of any two prime numbers is an even number, they may take some pairs of prime numbers and jump to make the generalisation that the sum of two prime numbers is even. Children should be helped to realise that *any number of examples cannot establish the truth of a general statement which must hold good for each and every instance and so one counter example will make it false.* For example, take the prime number 2 and add any other prime number and the sum is

odd. Similarly the statement that a quadrilateral having two diagonals cutting at right angles can only be a square or a rhombus, is not true. Consider Fig. 2.

This figure has no name and disproves the assertion by a counter example.

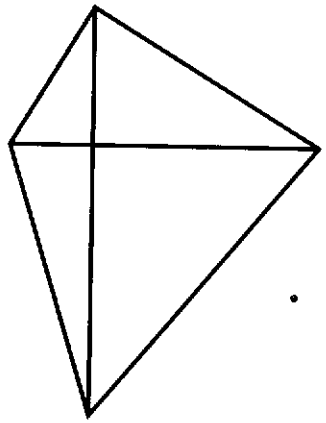
4. Proof by exhaustion

A perfect square number (in base ten numeration) always ends in double zeros, 1, 9, 5, 4, 6 and never 2, 3 and 7, 8. Children start with the statement that a number ends in anyone of the digits 0 to 9. So there are only ten cases to consider. Multiplying each of these digits by itself gives the digits that could occur in the units place of any perfect square. It is easily seen to be 0, 1, 9, 5, 4 and 6. Failure of digits 2, 3, 7 and 8 to appear in the units place shows that perfect squares never end in these. This is *proof by exhaustion*, since the digits that can appear in the units place of a number form a finite set.

It is necessary that an integrated view of mathematics is allowed to pervade the learning process at all levels of schooling. This can be done if children are exposed to all aspects of mathematics in smaller or greater measure as the situations permit and which the children can appreciate. This will help them avoid being obsessed with emphasis on computation and manipulation under the pretext of developing skills, as is by and large vogue now.

Atleast two of these are born on the same day. Do you agree?

Fig. 1



An Integrated View of Mathematics

Fig. 2

A9 – Mathematics with Two Graduated Rulers: 'Scales'

1. Prologue

Children love repetition but not monotony. They are happy when they get their insight stimulated through guidance and unhappy when they are simply asked to do things obediently as instructed. If this psychological trait of children is kept in view, mathematical learning can be made exciting and inviting.

In setting up addition situations to collect addition facts and build basic addition tables, children are introduced to the use of concrete objects and use of the fingers and then put under the regimentation of learning the tables by heart. This is a universally observed practice in primary schools. Learning by heart is not easy for many children and the effort required to do so is needlessly taxing. Most children consider the whole exercise a drudgery and recall of addition facts at random irksome if not painful. A better strategy will be to help children to gain familiarity with basic addition facts through suitable learning activities. A pair of graduated rulers (going by the name of 'Scales' in our schools, though not quite appropriate) offer immense advantage.

2. Two 'scales' for addition

Once children understand the meaning of addition not only with objects and fingers but also with sticks of different lengths and sticks of the same length; show them how to use two 'scales' to read off addition facts. One 'scale' is kept fixed and the other 'scale' is made to slide along the straight edge of the first. Initially the graduations on one scale are in alignment with those of the other, 0 with 0, 1 with 1, 2 with 2 and so on. Then zero of the sliding scale is brought into alignment with 3 of the fixed 'scale' and addition table 3 is ready to be read off (see Fig. 1).

For basic addition table of any specified digit number, the sliding 'scale' is moved along to have its zero in alignment

with the specified digit number on the fixed scale. Children should be required initially to explain addition facts by pointing out, as for example, that 3 intervals on the fixed scale (see Fig. 1) together with 2 intervals forward on the sliding scale give 5 intervals on the fixed scale and so on. The placement as seen in the same figure gives the subtraction facts:

10 – 7 = 3	6 – 3 = 3
9 – 6 = 3	5 – 2 = 3
8 – 5 = 3	4 – 1 = 3
7 – 4 = 3	3 – 0 = 3

Children should be helped to discover and realise that in case of subtraction facts, movement on the sliding scale is backwards (in opposite direction). They intuitively realise that *addition and subtraction are inverse operations* (see Fig. 2).

3. 'Scales' and fractions

Children's understanding of fractions is clouded as children are not presented the notions of a *whole and a part*, properly and adequately. *Whole and part are relative, there is nothing like the absolute whole and the absolute part.*

Children should be put through situations that would enable them to consider *anything as a whole or as a part. Anything can represent more than a whole also.* When anything represents more than a whole, the whole is contained in it and it is a good challenge for children to point out the whole in some special instances involving a sheet of paper in convenient shape or a piece of thread. Also when anything represents a part, the whole will be more than it and it is equally challenging to show how to make the whole.

A graduated ruler is quite handy in making the notions of a whole and part of it better understood. *Visualisation to start with promotes confidence in handling abstract ideas that emerge from manipulation of concrete objects.* By considering *two intervals* on a scale to represent *a whole*, children understand that *one interval* becomes *half* of the whole, three intervals *one and a half* of the whole and so on. By considering three intervals on the 'scale' to represent a whole, children

get to recognise a third, two thirds, one and one third, one and two thirds and so on (see Fig. 3).

4. Decimal fractions and 'scales'

Using the graduated ruler showing inches, with each inch seen divided into ten equal parts, decimal fractions can be identified. By taking 1 inch interval to represent a whole, children can point out, 0.1, (1/10), 0.2 (2/10) etc. and 1.1 (1+1/10), 1.2 (1+2/10) etc. Now by taking 10 inch interval to represent a whole, children can point out not only tenths but also hundredths. Step by step, children can be seen pointing out 0.01,0.02...0.10,0.11 ...0.19,0.20,0.21 ... etc.

5. Epilogue

How good it will be if each primary child is provided with a couple of graduated rulers as part of a mathematics learning kit.

Mathematics with two Scales

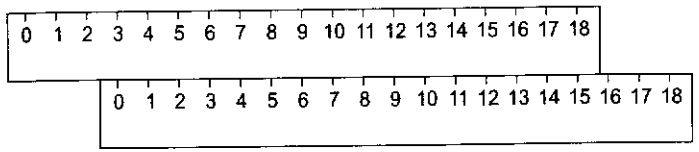


Fig. 1

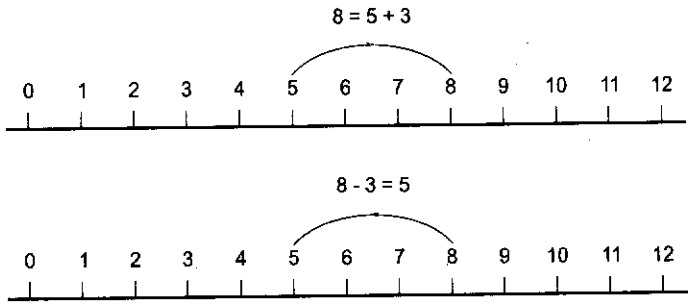


Fig. 2

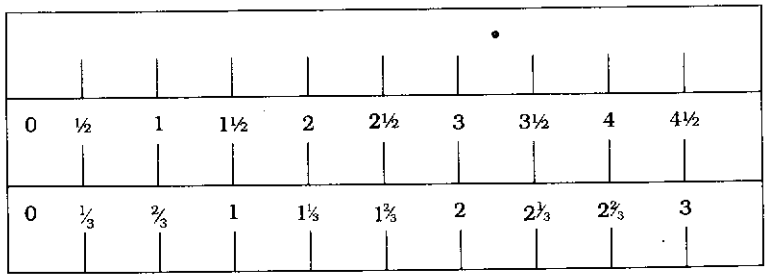


Fig. 3

A10 – Visual Aids for Multiplication Process

1. Prologue

Real learning in mathematics is exciting for children. If they want to get the thrill of mathematical thinking, they should experience the flow of thought process in passing from the concrete to the semi-concrete and from the semi-concrete to the abstract. Abstraction gives them the thrill and makes learning inviting and absorbing. Formalism resorted to early in the classroom teaching, robs this vital element in giving instructions, with the result that children fail to see any meaning in what they learn and gradually lose their natural flair for exercising their intuition. Some of the topics that are often seen to suffer from too early formalism are the operations with whole numbers in general and multiplication in particular.

2. Criss cross placement

Taking 18 broomsticks and placing a few of them one way and the rest the other way in a *criss-cross* placement (see Fig. 1) helps the children to grasp a basic multiplication fact. *Children build all the basic multiplication tables themselves by using this technique and in the process learn to memorise and recall all the basic multiplication facts from 0 × 0 = 0 to 9 × 9 = 81 with confidence and conviction. The approach is visual as children see the multiplicand, the multiplier, and the product separately* (Fig. 1),

This experience can be provide the basis for introducing a visual approach to help children do *multiplication of a multi digit number by a multi digit number.* Ten sticks when placed criss-cross with one stick indicate the multiplication fact 10 × 1 or 1 × 10 = 10 (number of junction points Fig. 2). Instead of ten sticks, ask children to use a thin cardboard strip to represent 10 (see Fig. 3) and place the strip criss-cross with one stick. Children easily recognise this to represent 10 × 1 = 1 × 10 = 10. The junction here represents 10. This is the passage to the semi-concrete or semi-abstract stage in learning process. Figure 4 represents two strips in criss-cross placement giving 10 × 10 = 100. Children learn to mark junction here as 100 (see Fig. 4).

The figures 6 and 7 show how to present multiplication of 32 by 24 by a *criss-cross visual.*

Children can read the junction values (see Fig. 7), count hundreds, tens and ones and pronounce the product to be 768 (see Fig. 7). They can also learn to do multiplication in two stages by taking first the criss-cross placement of 3 strips and 2 sticks representing 32 × 4 giving 128 (see Fig. 9) followed by taking the criss-cross placement of the same strips and sticks representing 32 × 20 giving 640 which on adding to 128 gives 768. *Had this kind of visual been incorporated in the teaching of new mathematics, many people would have had no hesitation to concede that the so called new mathematics might have had better chances to survive.*

Children see and learn to identify incidentally the *commutative* (5 × 3 = 3 × 5 from Fig. 1, 10 × 1 = 1 × 10 = 10 from fig. 2) and *distributive:* (30 + 2) × 4 = 30 × 4 + 2 × 4 and (30 + 2) × 20 = 30 × 20 + 2 × 20 from Fig. 7) properties of multiplication about which previously much fuss was made. For multiplication involving numbers consisting of more than three digits, strips of increasing thickness can be used to represent higher units 100, 1000 etc. Alternatively coloured strips with each colour to represent a specified higher unit may be used. Of course there would be no need to go beyond numbers with more than three digits, as children would have by then become sufficiently aware of the procedural pattern to discard this crutch.

3. Extension to decimal fractions

It is worth observing that, through this criss-cross placement, multiplication of decimal fractions can also be visually presented. This also serves as an excellent medium to read off some algebric identities, which to start with are simply generalisations of numbers and number operations. It is only a question of taking semi-concreteness to a higher level. This ensures that children have *a firm grip over their learning with the result that the reliability of their grasp of principles is high.*

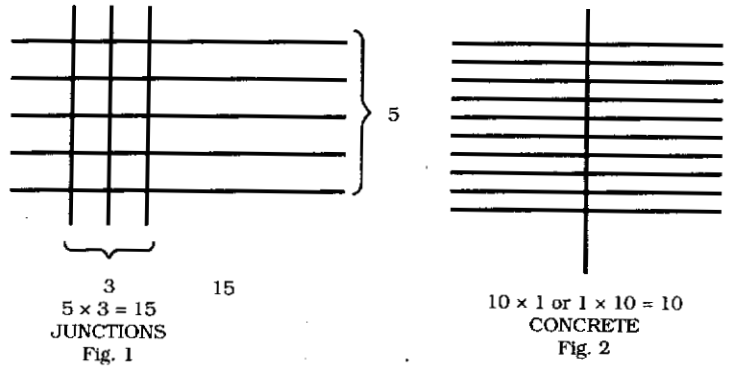
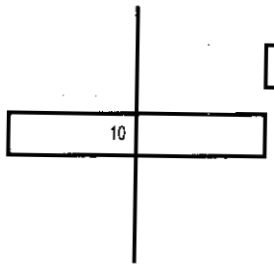
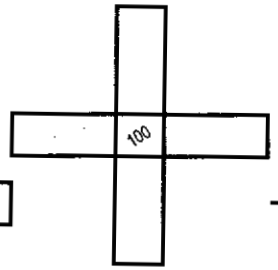


Fig. 1

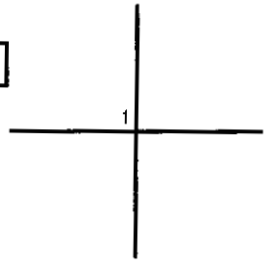
10 × 1 or 1 × 10 = 10
CONCRETE
Fig. 2



10 x 1 = 1 x 10 = 10
SEMI CONCRETE
Fig. 3



10 x 10 = 100
SEMI CONCRETE
Fig. 4



1 X 1 = 1
CONCRETE
Fig. 5

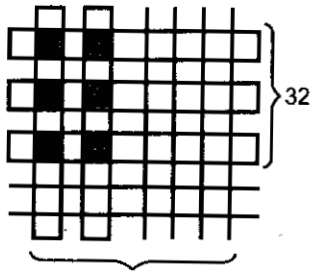
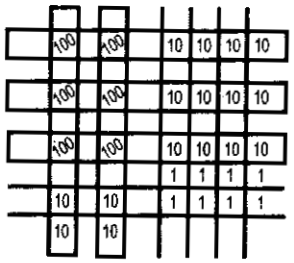


Fig. 6



SHOWING JUNCTION VALUES
Fig. 7

32	32
X 24	X 24
600	128
160	640
8	768
768	

Fig. 8 Fig. 9

A 11 – Instil Number Sense in Tiny Tots

1. Prologue

As soon as a child is old enough to remember and repeat what an adult says, parents and elders become impatient to test their children's memory. It is a widely prevalent practice among parents to teach tiny tots to count from one to hundred and pat themselves when the tots repeat the same from memory. When it comes to actual counting of objects children find the task irksome, as attention has not been devoted to the child's ability to match objects one by one with the ordered sequence of numbers. Some children become so sensitive that they develop a distaste for number work very early in their lives. How wholesome an effect it will have if this sort of thing is universally avoided!

Number is one of the most abstract concepts in mathematics and it has to be learnt in degrees. When one counts, the size, shape, colour, place, spacing, order, mass, capacity, time of existence etc. of the objects are not taken into account. If size is taken into account, the concept of measure gets developed. Therefore, this calls for patience in learning of numbers. The stages of learning synchronise with developmental stages in the growth of a child.

Instead of introducing *number names in their order*, it would be highly educative to have numbers up to five *introduced in any order*. This provides the child with the valuable opportunity of thinking and answering questions such as : 'What should we do to two to get three, to four to get five, to one to get two, and to three to get four?' This experience helps children discover 'one more' concept. They learn to say using fingers, *one and one more is two; two and one more is three* and so on. Now children state the numbers *one to five in order*. They also exhibit the urge to ask for names of higher numbers 'Five and one more is ...', 'Six and one more is ...' and so on *Every time 'one more' is said, a new number is got.*

2. Strategy for Evaluation

An important stage to evaluate arises at this stage. To test if the child has really grasped the number concept, ask the child to show in as many ways as he/she can, the same number of fingers as you show (see figures 1 and 2 for showing three and two in different ways).

At first, children, find it challenging but soon intuition helps them to accomplish the task with an air of triumph and confidence to the astonishment of all. The child invites respect as it does so untutored. Individual differences among children also surfaces, distinguishing the fast learners from the slower ones, incidentally, this test gives children a lesson of vital importance and develops in them readiness for doing addition and subtraction, multiplication and division, giving additive complements of a number etc.

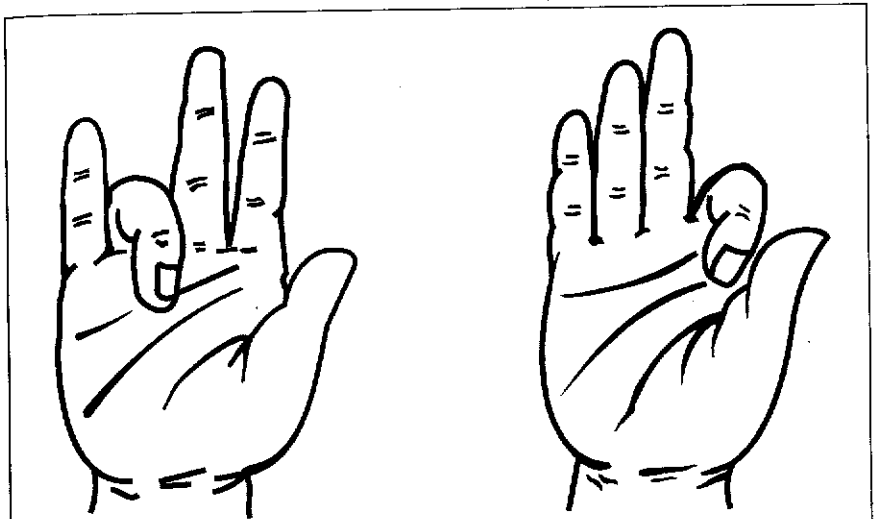


Fig. 1

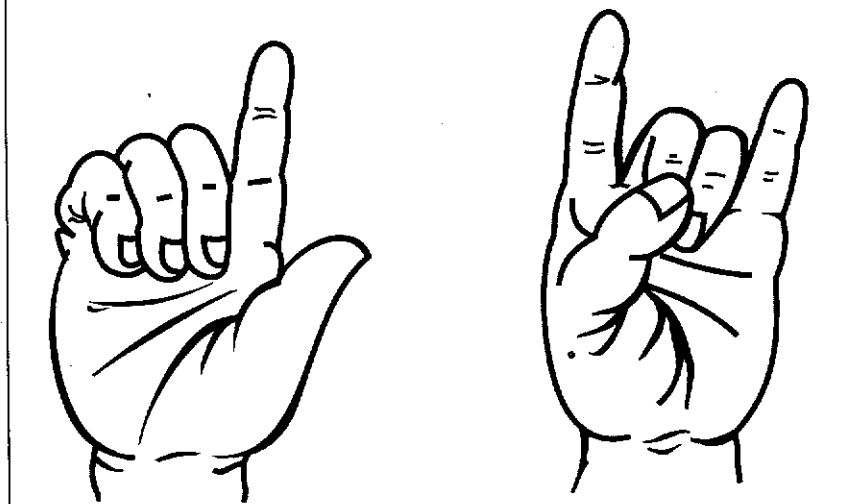


Fig. 2

A 12 – Conversion of Numeral Base on Fingers

1. Prologue

Everyone is blessed with fingers, that can be put to imaginative use in explaining and understanding many concepts in school mathematics, besides counting and computing. With the advent of the computer, children are taught nowadays the numeration to different bases, particularly bases 2 and 5 and conversion of numbers from one base to another. Conversion of numbers from base two to base ten and vice versa naturally receives attention in school mathematics as well as in computer science.

Fingers come in handy to convert base two numerals into base ten numerals. When children see the possibility of finger conversion for bases other than two. They can go up to base four.

First of all, children can be easily helped to realise that the number of digits or basic numerals in a base system of numeration, is the same as the number of the base of the system. For example, the digits in base ten numeration which is the most common and familiar one are ten in number and they are 0, 1,2, 3,4, 5,6, 7, 8 and 9. For base nine numeration the digits are 0 to 8 and they are nine in number. Also for base 8, they are 0 to 7 and so on. Finally in base two numeration the digits are two in number and they are 0 to 1.

It can, therefore, be seen that base two numeration on fingers is the simplest as only two positions corresponding to the two digits 0 and 1 have to be set up and identified. *If a stretched finger is associated with* 1, *then a bent finger will represent 0.*

2. Strategy

Ask the child to keep its right palm with its lines facing him/her. Assigning the place values for the fingers starting from the thumb, *the thumb can be taken to represent one, the pointing finger two, the middle finger four, the ring finger eight* and *the little finger sixteen.* Some children may like to write the values on their fingers in ink. When a finger is bent, the value of the finger gets omitted (see Fig. 1).

The child feels delighted to reel off base two numerals and their corresponding base ten numerals just by looking at the fingers some of which will be stretched and the rest bent. With five fingers, the numbers up to 31 can be read off in base two numeration. The pictures here illustrate some cases.

10101 two = 21 ten (1 + 4 + 16) in Fig. 2

11100 two = 28 ten (4 + 8 + 16) in Fig. 3

If the left palm is also used and the fingers given the place values beyond sixteen in base two numeration, that is 32 (little finger), 64 (ring finger), 128 (middle finer), 256 (pointing finger) and 512 (thumb), then numbers up to 1023 (in base ten) can be converted to their equivalents in base two system.

3. Project

Following base two representation on fingers, children would be curious to look for setting up finger positions to represent base 3 and base 4 numerals. Base 3 needs three positions and base 4 four positions. Once they hit upon positioning each finger in three ways as well as four ways, children are ready to read off numbers in base three and base four numeration. To go beyond base four becomes extremely cumbersome and hence difficult and so is not tried at all. Using both the palms, addition and subtraction of four digit numbers in base two numeration can be done by children with little instruction.

4. Instant sum

Incidentally children can be helped to discover that the sum of all the powers of two up to any index is simply one less than the power of two to follow the last index. That is to say

$1 + 2^1 = 2^2 - 1$

$1 + 2^1 + 2^2 = 2^3 - 1$

$1 + 2^1 + 2^2 + 2^3 = 2^4 - 1$ and so on.

Note

But children will have to be careful in computing the conversion in bases greater than two, as it is not simply addition of values as in base two but addition of multiples of values in other bases. When in any system the thumb of the right palm can be taken to carry the place value 1, the pointing finger, the middle finger etc., will carry places values three, nine etc. in base three and place values four sixteen, sixty four etc. in base four and so on.

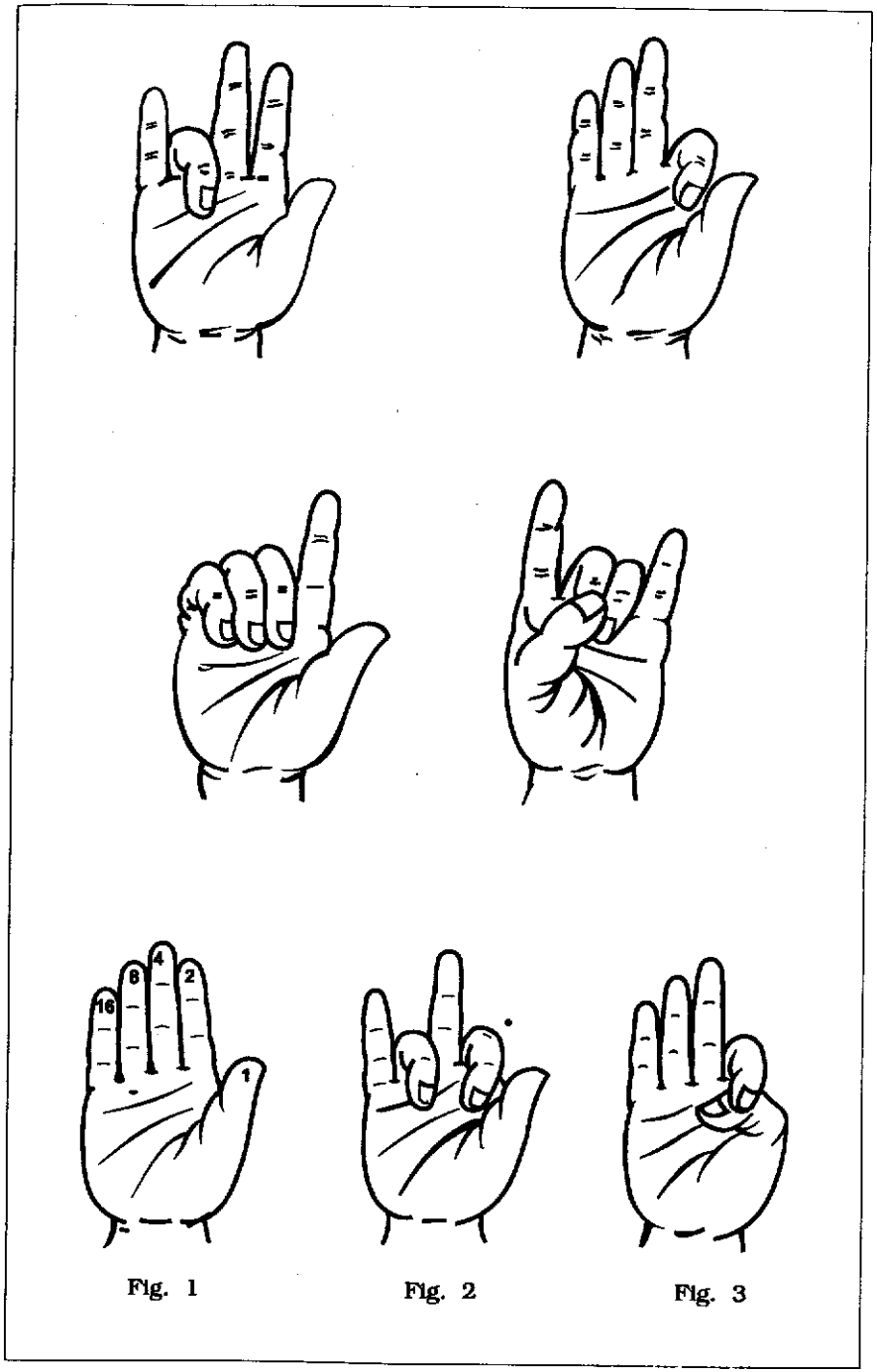


Fig. 1 Fig. 2 Fig. 3

A 13 – Number Formations

1. Prologue

Spat intuition that children possess is not fully utilised in their schooling, particularly in mathematics, resulting in an interglobal imbalance in the learning process. Once children learn to count up to ten and put objects in a row, they can be exposed to odd and even, and triangular and square number formations.

2. Strategy

Odd number formations are presented through two rows, one showing a number and the other its predecessor. Children can be observed developing initiative to set up any odd number, given its position in the sequence shown below:

Odd numbers	1st	2nd	3rd	4th	5th

Similarly any even number can be set up in two rows, each giving the same number, as shown:

Even numbers	1st	2nd	3rd	4th	5th	6th

Triangular numbers can be set up in sequence, as shown below:

Triangular numbers	1st	2nd	3rd	4th	5th

Next children can set up square formations as shown below:

Square numbers	1st	2nd	3rd	4th

3. A variation

Instead of setting up odd numbers in the form of two rows to show a number and its predecessor, they can be presented in a row and a column as illustrated below:

Odd numbers	1st	2nd	3rd	4th	5th

This provides the child an opening to build a square number with consecutive odd numbers as shown:

1st 2nd 3rd 4th 5th

Square numbers

1	3 + 1	5 + 3 + 1
1st odd no.	+ 2nd odd no.	3rd odd no.
	+ 1st odd no.	+ 2nd odd no.
		+ 1st odd no.

7 + 5 + 3 + 1 +
4th odd no.
+ 3rd odd no.
+ 2nd odd no.
+ 1st odd no.

By setting up two consecutive triangular number formations, one upright and the other inverted, children can discover that any two consecutive *t* (triangular) numbers together form a square number. See the illustration below:

All these can be presented 'live' on the playground or on the stage by children through taking positions with background music.

The rule may not be found easily or *at all* and the pattern (or may be the table itself) would suffice as illustrated below.

Game 3

Leela	(x)	3	7	8	6	2	5
Lata	(y)	5	2	5	2	5	2

3. Three children games

In the three children game, two of them give a number at random and the third relates his number to those given by the others. This goes on till the audience joins in.

Game 1

Raju	(x)	3	7	1	3
Rana	(y)	5	6	4	8
Ray	(z)	8	13	5	11

The audience discovers that what Ray says is the sum of the numbers given by Raju and Rana. In other words $z = x + y$ or $f(x) = x + y$.

Game 2

Girish	(x)	1	2	4	7
Gopal	(y)	3	5	6	7
Giridhar	(z)	10	29	52	98

Here the audience gropes for a while and then discovers that what Giridhar says is the sum of the squares of numbers given by Girish and Gopal. The intricacy of this game is greater and hence provides more excitement.

The children should be helped to realise that a number and its multiples bear functional relation. There is another important point that should not be lost sight of. A functional relation need not always be associated with numbers. For example, consider a gathering of mothers and their children. The relation that each child bears to a mother is functional.

Thus, the concept of functional relation can be easily and effectively brought home to children through playground experience, placement games and stage presentations.

A 15 – Function Concept Game - II

1. Prologue

Formalism often hinders conceptualisation in mathematics. Conceptualisation can be facilitated over the years if experiences are provided at different levels in stages. Function concept is central and crucial to sound acquisition of mathematical knowledge.

A good start for function concept can be made even at the kindergarten stage. 'One-way see-one game' and 'Two-way see-one game' give a good start, as they are within the capability of five-years old children.

2. One-way see-one game

Set up a network of strips showing say, five rows of five junctions each; place an object, a bottle top, at each junction. Ask the child to leave only one object in each up-down set (or column) and remove the rest so that only one object is seen in each up-down set. This can be played 'live' with children standing in five rows of five columns each. As the responses are numerous, this demands exercise of imagination and spatial intuition, thereby promoting creativity.

'Both ways see all' arrangements (see Fig. 1).

3. Two-way see-one game

After children get settled in playing this game, introduce 'Two- way see-one' game. A child should ensure that only one object is seen not only in each up-down set but also in each left-right set. Fig. 2 satisfies this double criteria but not Fig. 3 and 4.

Later in higher classes, children can be asked to name each junction by means of an ordered pair of natural numbers, giving 'the place number' and the 'the row number' of each junction. If children name the points in 'One-way see-one' arrangement, they find that no two ordered pairs have the same first number (or component), though the second number

may be the same or different. In Fig. 2, the marked junctions can be listed as (1,4), (2,3), (3,5), (4,2), (5,1).

In fig. 3 the listing is (1,3), (2,5), (3,3), (4,5), (5,2), and in Fig. 4 it is (1,2), (2,2), (3,2), (4,2), (5,2).

4. Preview

This experience provides the basis to suggest and appreciate the vertical line test and the horizontal line test in the case of continuous line graphs later, if the graphs are to represent functions. A learner will now be in a position to give the formal definition of a function as 'a set of ordered pairs no two of which have the same first component' and if, in addition, no two ordered pairs have the same second component, the inverse of the *function* is also a function.

'Both ways see all' arrangements

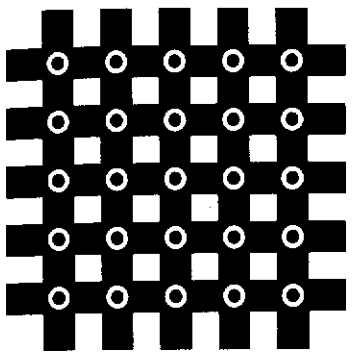


Fig. 1

'One way see one' arrangements

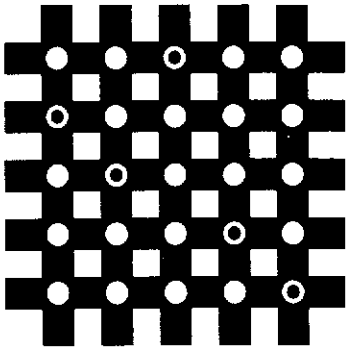


Fig. 2

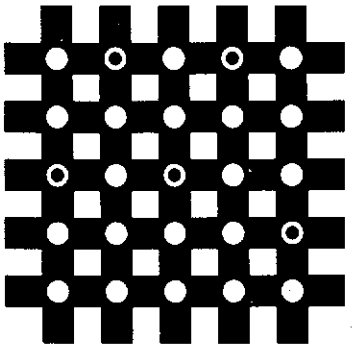


Fig. 3

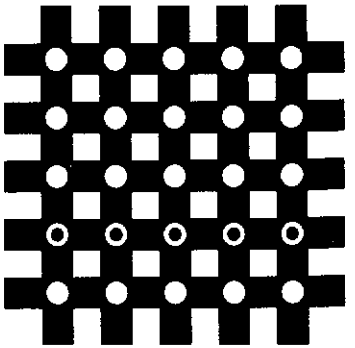


Fig. 4

B. Experiences with Shapes

B 1 – Mathematics with Railway Tickets

1. Prologue

Mathematics can be interpreted as a way of looking at things and their set ups and discovering relations among them.

Give the children four railway tickets of the same size. Tell them to arrange the tickets in such a way that the whole arrangement is square in shape with a hollow square inside. Thus there are two squares one inside the other. After some trial and error, it is easy for the children to arrive at the solution as shown in Fig. 1.

2. Setting up design

Children know what a rectangle is. Railway tickets are usually rectangular. If they know that the area of the rectangle can be given as ab; with a and b representing respectively the measures of its length and breadth, they can translate the design into a beautiful algebraic identity. At first, they will have to examine the design and discover that

(1) the outer square without the inner square gives four rectangles of the same size,

(2) the side of the inner square is seen as the difference of length and breadth or a–b. Then the design gets easily translated thus:

$$(a + b)^2 - (a - b)^2 = 4ab$$

Next let the children draw the diagonal joining a pair of opposite corners in each rectangle as shown in Fig. 2.

What is the design that is seen now? It is a three square design, and there are eight right-angled triangles of the same shape and size. The third square is sandwiched between the outer and inner squares seen earlier. Now the students can discover a property of the *sandwiched square*.

The outer square without the four right-angled triangles (of the same shape and size) gives the sandwiched square. In other words, the outer square less four right-angled triangles, each of which is half the area of rectangle, gives the square on the longest side of any of the right-angled triangles. The side of the outer square is seen to be $a + b$ and let the longest side on any of the right-angled triangles measure, say, 'c'. Now the sandwiched square design can be translated into symbols as follows (see Fig. 3) :

$$(a + b)^2 - 4\ (1/2\ ab) = c^2$$

This on simplification gives $a^2 + b^2 = c^2$ which is nothing but the famous Pythagorean property of a right-angled triangle. Incidentally, this is the famous 'Behold' demonstration of Bhaskaracharya.

There is also another way of looking at the sandwiched square (see Fig. 4). The inner square together with the four right-angled triangles (of the same shape and size) gives the square on the longest side of any of the right-angled triangles.

This when translated into symbols gives $(a - b)^2 + 4\ (1/2\ ab) = c^2$ which on simplification again gives $a^2 + b^2 = c^2$

Interestingly enough, the removal of the four right-angled triangles from the outer square can be presented using only two tickets as shown in Fig. 5. Remember that each rectangle consists of two right-angled triangles.

This design shows that the outer square less four right-angled triangles gives two squares, one on one of the two smaller sides and the other on the other smaller side of any of the right-angled triangles. This finding coupled with the one seen earlier with outer square and sandwiched square gives the well-known property that the sum of the squares on the two smaller sides of a right-angled triangle is equal to the square on the longest side of the right-angled triangle.

This is the simplest and the most elegant Chinese proof not requiring the use of algebraic expressions.

Use of four copies of a visiting card or a post card would be beneficial by way of variation in enriching the experience.

3. A Project

To find out if the children have developed an insight in understanding the property of a right-angled triangle, pose the problems suggested below:

1) Fix up a square which is equal to the sum of two given squares of different sizes.
2) Fix up a square which is equal to the difference of two given squares of different sizes. (Note: Equality in the above is in terms of areas).

Give children two thin cardboard squares of different sizes and tell them that the solution does not need any measurement or calculation. It is enough if a method of solution is just described.

Note: It is necessary to warn them that the use of objects in the mathescope is only to help visualisation of plane figures as shown in diagrams.

Maths and Railway Tickets

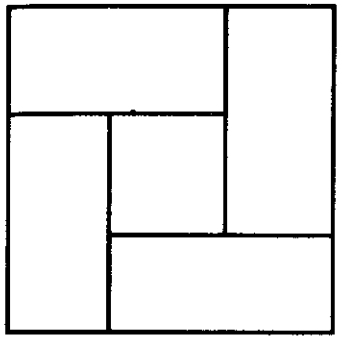


Fig. 1

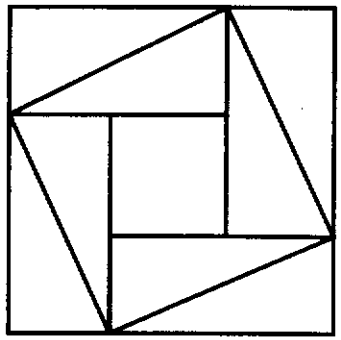


Fig. 2

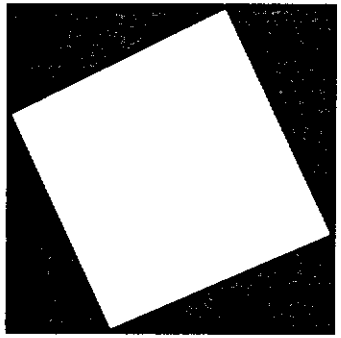


Fig. 3

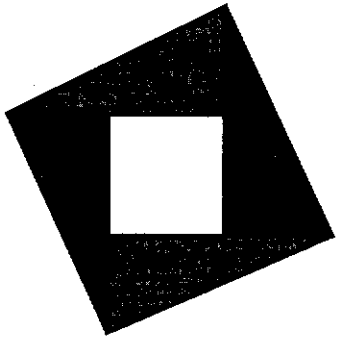


Fig. 4

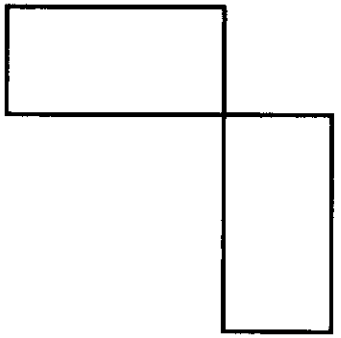


Fig. 5

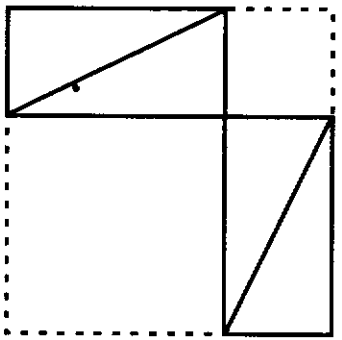


Fig. 6

B 2 – Not a Game of Chance

1. Prologue

A thoroughfare in a town. A person along with one or two of his assistants chooses a spot on road-side, spreads a towel, places on it a wound-up belt showing two gaps, inserts a pencil in one of the gaps and calls for betting to say if the pencil is inside or outside, with the promise to give any person responding correctly, double the amount advanced by the latter as the bet.

One of the assistants poses to be a passerby and is seen to bet one rupee, give a *correct* response and carry away two rupees. This attracts and entices a few passers-by who stop to be onlookers. Soon the craze for making a quick buck grips them and they join the betting game. After about half an hour, the gambling den is vacated to move to a new spot on some other thoroughfare.

Very few in any town would have failed to notice this, at one time or other, and might have betted just to try their luck.

2. Strategy

How is a belt wound up to show two gaps. Take a long belt, fold it about the middle, hold the overlapping parts together and wind it round and round from the folded edge till the free ends are reached. The two gaps will be formed. Press the free ends with one hand while holding the roll and insert a pencil in either of the two gaps with the other hand to ask if the pencil is inside or outside. Once the response is made, allow the belt to unwind itself by holding the pencil and pulling the free ends. When the pencil is caught by the folded belt, the pencil is said to be inside, when it is not caught it is outside (see Fig. 1, 2, 3).

3. A game of chance

Is this really a game of chance? The investigation requires making of a mathematical model of the game with the belt by ignoring the non-essential features. An intermediate step is to use a cord instead of a simple closed curve.

What is a simple closed curve? Start from some point and draw a curve in such a way that you come back to the starting point without lifting your pencil or pen. This describes, by and large, a simple closed curve. A circle and a square are examples of a simple closed curve. A simple closed curve drawn on a plane has an inside and an outside. This is quite plain when the curve has no convolution (see Fig. 4). But when the curve is highly convoluted, it baffles one to locate a point in its inside or outside. The figure 4 is an example of a simple closed curve.

There is a beautiful test to solve the problem of determining if a point is inside or outside a simple closed curve. Count the number of intersections.

In figure 6, I is a point inside the closed curve. Two rays from I are drawn and the intersections of each with the curve are counted. The number of intersections of the ray IX in the figure is three and those of IY is five. Let children draw other rays from I and count the intersections of each with the curve. The number will every time be found to be odd.

O is a point outside the closed curve (see Fig. 5). Two rays to intersect the curve are drawn from O and the number of intersections of the ray OP with the curve is 2 and that of OQ is 4. Let children draw other rays from O to intersect the curve and count the intersections of each with the curve. Children discover the number to be even. They enjoy the fun of drawing simple closed curves as highly convoluted as possible, and determining their insides and outsides by means of the ray test.

So to decide if a point is inside or outside a simple closed curve, what one has to do is simply to draw or imagine a ray from the point to intersect the curve and count the number of intersections of the curve; if the number turns out to be odd the point is inside and if it is even, the point is outside (see Fig. 5).

Now consider the closed curve which models the game with a belt or cord.

N and W are two points. We have to determine which of the two is inside the closed curve. Draw a ray from each cutting the curve. NT is a ray from N and WS from W. Let us count the intersections of the curve with each ray. NT has five intersections and WS four. So N is inside and W outside closed

the curve. You can also check by tracing your way out from Z (see Fig. 7).

Instead of drawing the ray, one can simply and speedily count in twos or fours the portions of the curve adjoining the gap containing a given point. *If the number of 'portions' is odd, the given point is inside the curve and if the number of portions is even, the given point is outside the curve.*

4. Epilogue

As a game of chance, this betting game with the belt has some glamour. But by applying the profound but simple property of a simple closed curve on a plane which says that it has an inside and outside, a property named after the mathematician Jordon who was the first to give a formal proof of it, the chance element in the game is removed and the unfailing rule to win the game always is discovered, assuming that no attempt is made to cheat.

Not a Game of Chance I

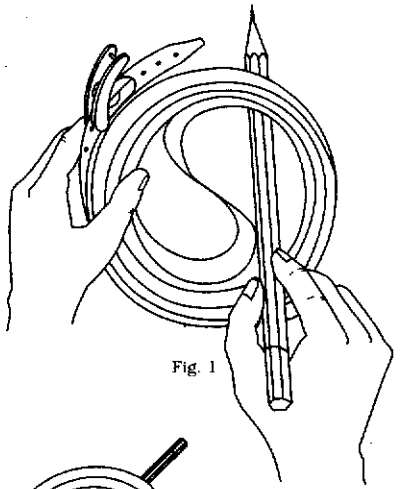


Fig. 1

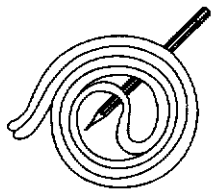


Fig. 2

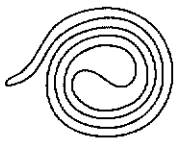


Fig. 3

Not a Game of Chance II

Fig. 4

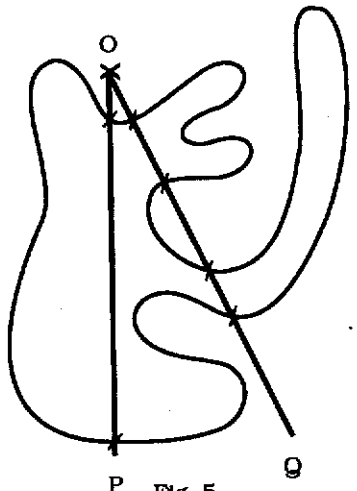


Fig. 5

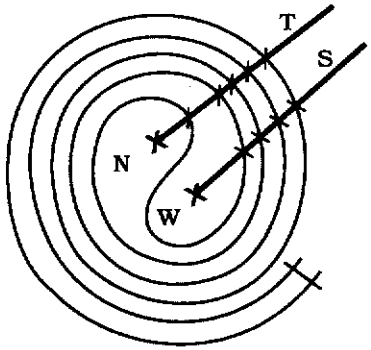


Fig. 7

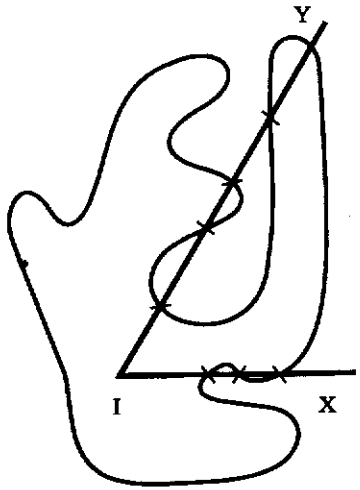


Fig. 6

B 3 – Mathematics with a Ruled Sheet

1. Prologue

All structured situations have beautiful mathematical properties imbedded in them. One such is the ruled sheet of paper, easily available to everybody.

A ruled sheet with a line of margin on it (usually on the left) is preferable. If the margin is not there, roll the sheet lengthwise for some distance from the right edge so as to have some space as margin and draw the marginal line. It is assumed that the rulings on the sheet taken are at equal intervals or equidistant. It is advisable to check this assumption first by cutting off a narrow lengthwise strip of uniform width on the right side of the sheet and by pushing it up and down the rest of the larger part of the sheet for testing of line or interval alignment. Also the line of margin should be vertical when the sheet is held up so as to have the ruling horizontal position. The closer the rulings are, the better the scope for explorations.

2. The Ruled Sheet

The first thing that a child can easily observe is the equality of intercepts made on the line of margin by the rulings. The child can also see that the line of margin makes with each ruling a square corner, i.e., a right angle.

Draw a line with its ends on any two rulings which are not consecutive (see Figs. 1a and 1b).

Using a pair of dividers, compare the intercepts made on the line by the rulings. What do we discover? The intercepts are all of equal length. Draw many more line segments and verify if equality of intercepts occurs in each case. This experience is the basis of one of the most significant theorems viz. the *Intercepts Theorem*, in elementary Euclidean Geometry.

The line of margin cuts the rulings. From the point where the line cuts a ruling, draw any other line segment with its other end on some other ruling. One can observe a sequence of triangles being formed (see Fig. 2a).

What kind of triangles are these? The angles of all the triangles measure the same. This can be verified by taking another set of triangles (made by using a carbon sheet) and superimposing the angles of one over the angles of another, one by one.

The corresponding sides of any two of the triangles, are however, in the same ratio. In other words, if any two triangles of the sequence are examined, one is a reduction or an enlargement of the other.

Incidentally the children find it worthnoting that given *three angles of a triangle, no unique triangle is found.* On the other hand, numerous triangles exist. But it is not so, *when three sides of a triangle are given, in which case a unique triangle is determined.*

Triangles which have their angles alone preserved are called *similar.* Their corresponding sides are *proportional.*

See the diagram below (Fig. 2b). The bases of the triangles in the sequence are in the ratios of 1 : 2 : 3 : 4 : 5, as is the case with the vertical sides if they are drawn or the slant sides of the triangles.

Looking at the sequence from the other end, a child can see that the bases of the triangles can be seen to be

1, 4/5, 3/5, 2/5 and 1/5 (see Fig. 2b).

This opens up a method of getting the parts of a line segment using a ruled sheet with a line of margin. Drawing verticals to the base from the points that the slant line segment makes with the rulings, a child can see that the base is divided into five equal or congruent parts. This suggests the use of a ruled sheet to divide a line segment into the required number of equal parts (allowing for limitations imposed by the size of the sheet).

3. Parts of a line segment

An interesting question arises. Is it possible to get parts of a line segment when a ruled sheet without the line of margin is used? First of all let us understand the steps and outcome of the previous experiment.

We said the line segment which was to be divided along one of the rulings beginning from the margin. We counted five rulings above the line segment and joined the point of

intersection of the margin line with the fifth ruling to the other end of the line segment. From the intersection points of this slant line-segment with the ruling perpendiculars were drawn on the given line segment. Thus we got
1/5, 2/5, 3/5, 4/5 parts of the segment.

The ruled sheet to be used now has no line of margin. The first thing to be done is to fix the line segment of given length along any ruling. Suppose sevenths of the line segment are to be obtained. What is the technique that should be used ? (see Figs. 3a and 3b).

Count seven rulings above the line segment and *take a point on the seventh ruling.* Join the point to the end points of the line segment. A sequence of similar triangles is formed again. Use a pair of dividers and check that seven times the base of the smallest triangle is equal to the line segment.

That is to say, the base of the smallest triangle is 1/7 of the line segment which happens to be the base of the biggest triangle, the length of the base of the next bigger triangle is 2/7 of the line segment and so on.

If parallels to one of the sides are drawn from the points of intersection of the other side to meet the base, the base is seen to be divided into seven equal or congruent parts.

Mathematics with a Ruled Sheet

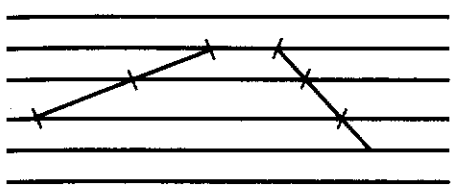


Fig. 1a

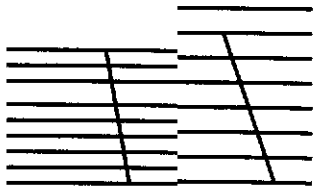


Fig. 1b

Maximum number of intercepts of a line segment. Notice more intercepts on a sheet with closer rulings.

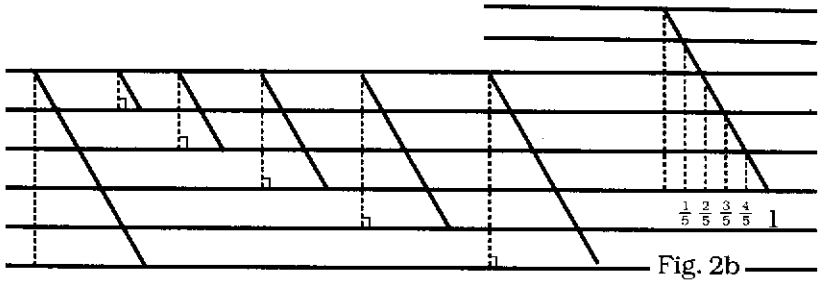


Fig. 2a

Fig. 2b

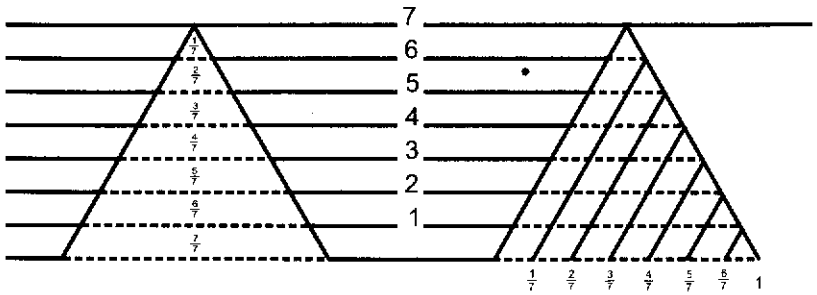


Fig. 3a

Fig. 3b

B 4 – Explorations on a Ruled Sheet

1. Strategy

Take a point on a ruling, say, in the middle of the sheet. With that point as centre, draw a circle of convenient radius. Join the centre point to the points of intersection of the circle with the rulings. What do children notice about the radii of the circle? The radii are divided into halves, thirds, fourths and fifths in the same diagram. Is this not fascinating? (see Fig. 1)

An instant method of dividing a line segment into the required number of equal parts suggests itself at this stage, provided *the line segment is longer than a segment consisting of equal number of intercepts on the line of margin.*

Choose an appropriate segment of intercepts (i.e., with as many intercepts as the number of subdivisions required) on the margin line. With bottom end of this segment as centre and line segment to be divided as radius, draw an arc using compasses. Let it cut the ruling passing through the other point of the marginal line segment.

Any other line segment required to be divided into the same number of equal parts can also be fixed up on similar lines as shown in the diagram (see Fig. 2).

The ruled sheet comes handy in exploring the relationship between the diameter of a circle and its circumference.

2. Innovative approach in finding II

Use cardboard or plastic discs of convenient sizes. Take one of them and place it in the left corner space to touch the margin line and the first ruling. Trace its rim on the ruled sheet. Lift the disc and now place it touching the first rim outline and the top ruling as shown in Fig. 3 below Trace its rim once again and repeat it four times. Through the points of contact of rim outlines draw a line as shown. This is the line of diameters. Mark the diameter distances as 0, d, 2d, 3d, 4d as shown in Fig. 3.

Make a mark on the rim of the disc. Hold the disc upright on the line of diameters matching the mark on zero of the line

of diameters. Roll the disc very carefully along the line of diameters without sliding or slipping and watch for the mark. Once the mark is back on the line of diameters stop rolling the disc. Mark this point on the line of diameters. *The distance covered by the disc can be taken to be its circumference.* Repeat the rolling to ensure accuracy as far as possible.

A child doing this interesting explorative exercise discovers that *the circumference of a circle is a little more than three times its diameter.* The question that naturally arises at this stage is, 'what part of the diameter is the excess portion beyond 3d'?

Use dividers and with its legs spread to cover the excess portion correctly, step off to find the number of such steps to cover the diameter from 3d to 4d. If stepping is done very carefully, it is seen that the *excess portion is roughly a seventh of the diameter.*

In figure 3, the base of the first triangle from the bottom most vertex is almost equal to the excess portion. This shows that the circumference of a 'circle is plausibly 3 1/7 times its diameter. Actually, the relationship cannot be expressed in terms of a fractional number, though the ratio is a constant denoted by π, the Greek letter corresponding to *p*, standing for perimeter and pronounced 'pie'.

Repeat the exploration with discs of other sizes to ensure that circumference = 31/7 is the property of all circles.

What is noteworthy about this innovative approach is that *no scale is used in measuring the diameter and the circumference and in finding their ratio,* as is usually done. Also the approximate value of π viz. 3 1/7 is actually obtained by the child without being told.

3. Forming quadrilaterals

Choose three straight broomstick pieces, two of which are of the same length. Make a square corner with paper. Use the ruled sheet to mark the middles of the broomsticks.

On a ruling in a ruled sheet, choose a point. Through that point place two broomsticks of unequal length in such a way that there middles lie on the point. Mark the ends of the broomsticks and join the marks consecutively and identify the quadrilateral formed. Now a parallelogram is

formed (see Fig. 4a). When the broomsticks are placed so as to have a square corner between them, a rhombus is formed (see Fig. 4b).

When the middle of only one broomstick lies on the point and the other broomstick is so placed through the point as to make a square corner, a kite is formed (see Fig. 4c).

When the two broomsticks are so placed that they lie across each other meeting at some point on the ruling, a quadrilateral is formed. If in addition, the broomsticks make a square corner, a quadrilateral with its diagonals cutting at right angles is formed (see Figs. 5a and 5b).

Now take two broomsticks of equal length. Place one across the other such that their middles lie on the point chosen on the ruling. Mark the ends of the broomsticks and join the marks consecutively. What kind of a quadrilateral is formed? It is a rectangle. If the broomsticks make a square corner at their meeting, the quadrilateral is a square (see Figs. 6a and 6b).

Place two broomsticks across each other in such a way that the top end-points of each broomstick lie on some ruling and bottom end-points lie on another ruling. Joining the end-points of the broomsticks consecutively, a trapezium is seen to be formed. If the broomsticks are of equal length, then the trapezium formed is isoceles (see Figs. 7a and 7b).

Explorations on a Ruled Sheet I

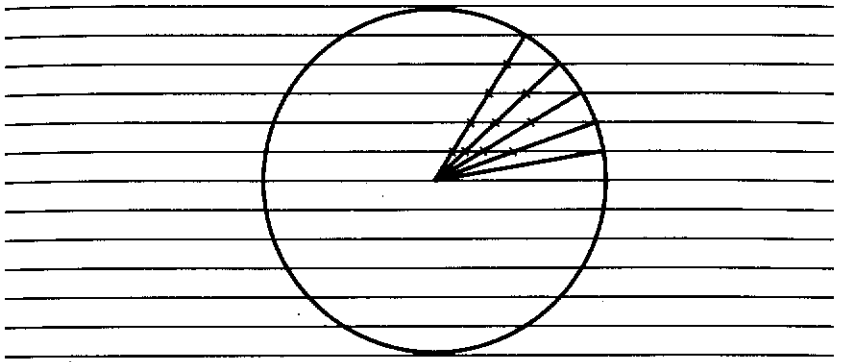


Fig. 1

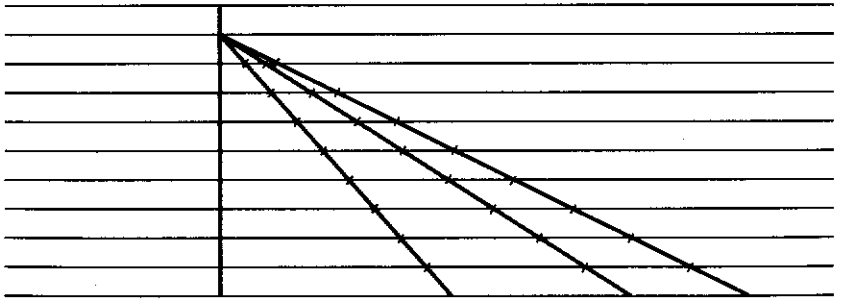


Fig. 2

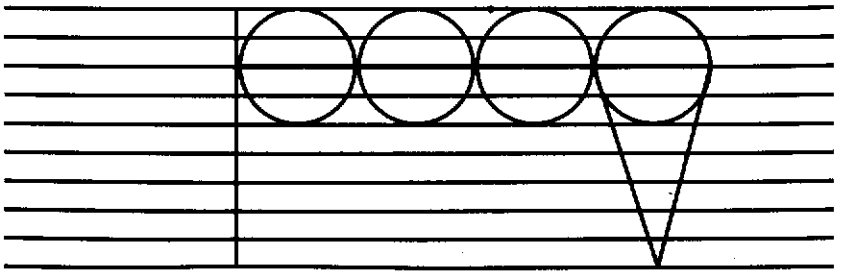


Fig. 3

Explorations on a Ruled Sheet II

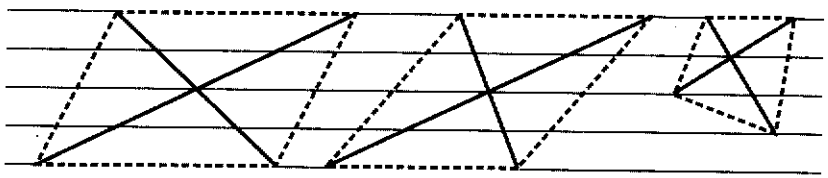


Fig. 4a Fig. 4b Fig. 4c

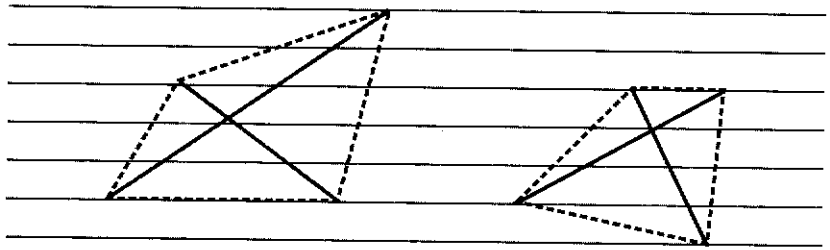


Fig. 5a Fig. 5b

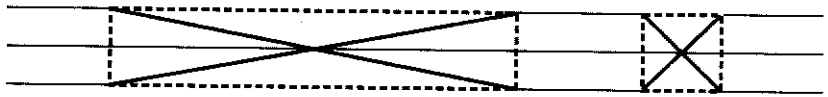


Fig. 6a Fig. 6b

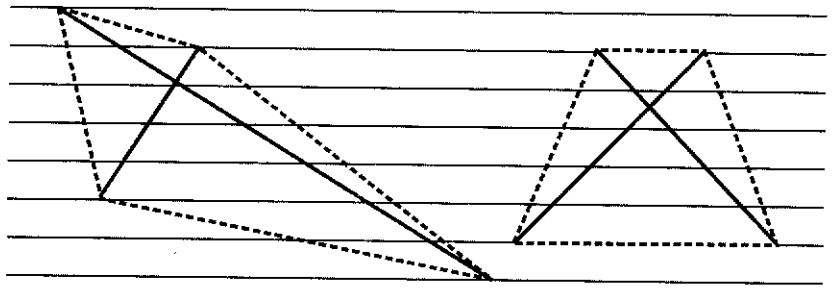


Fig. 7a Fig. 7a

B 5 – Checkruled Areas

1. Prologue

A checkruled or a square ruled notebook is widely used in schools for number work, as it helps in avoiding problems of place value alignment. Potentialities of a checkruled sheet for mathematical investigations by children have neither been explored nor exploited by the majority of teachers. It would be highly educative if the teachers in their pre-service or in-service training schedules are made thoroughly familiar with this avenue of mathematical learning. The versatility of a checkruled sheet is so immense that it can be made use of effectively at all levels of schooling in mathematics.

2. Area and perimeter

Area and perimeter of a figure are expressed in different units and that is why they cannot be compared. Only their numerical values can be compared. An interesting project for investigation is about the numerical comparison of area and perimeter of geometric figures. A square ruled sheet that is commonly available is quite adequate for this purpose. To start with, let us deal more with rectilinear figures, that is, figures bounded by line segments.

When mathematics is taught formally and is devoid of any experimentation or discovery, it generates numerous misconceptions in the minds of learners. *Unless children are actively encouraged to play around with an idea, it is very likely that they may reach an incomplete or faulty understanding.*

Ask a child in middle school / upper primary, What shape does a figure of area 16 sq. cm have? A majority of children, exposed as they are to learning by imitation of set models and acceptance of ideas which are narrated, can be seen to answer unhesitatingly that it is a square, when pressed to think if it was the only answer, some manage to say that it can be a rectangle. But very rarely can one come across

children answering that it can have any shape. The checkruled sheet can be used to grasp this important idea by converting a square (or a rectangle) into different shapes without changing the area, as shown in Fig. 1.

3. The same area in different shapes

Showing the same area in different shapes is a creative exercise providing a valuable mathematical experience. This project enables the children to feel that, given a figure of a particular shape (closed plane figure to start with), it has a definite area, but given the definite area of a figure, the figure has no particular shape.

The next question that can be taken up for investigation is the numerical comparison of the measures of area and perimeter of rectilinear figures. Ask the children to build rectilinear shapes of different kinds having the same area, say of six square units, on a checkruled sheet and compute the perimeter of each shape. A sample work done by children is given in Fig. 2.

4. The Questions

Incidentally children stumble on the discovery that *figures can have different shapes not only when their areas are the same but also when their areas and perimeters are the same.* The investigation gives children the experience to answer more questions such as :

(1) Can the perimeter of any rectilinear figure be numerically equal to its area?

(2) Can the perimeter of any rectilinear figure be numerically less than its area?

(3) Can a rectilinear figure of given area be drawn to have the required perimeter and vice versa?

While trying to answer such questions, children come up with certain interesting findings. *Of all the rectangles which can be built to have the same perimeter, the square has the maximum area.* This problem is, interestingly enough, seen to be related to the question of finding among pairs of numbers having the same sum, the pair with the greatest product. Take,

for instance, pairs of numbers having the sum 10. (1, 9), (2, 8), (3, 7), (4, 6), (5, 5), when counting numbers alone are considered. Finding the product of each pair one can easily see that when the pair of numbers are equal, the product s the greatest.

5. Versatility of square ruled sheet

This same project when handled by high school and higher secondary pupils can be seen to demand use of higher mathematics embracing irrational numbers, quadratic equations and differential calculus. If instead of rectangles, other polygons and circles are included, the understanding that given a piece of rope, the maximum area that can be measured out by bringing the two ends of rope together should be circular in shape, would be secured.

Checkruled Areas

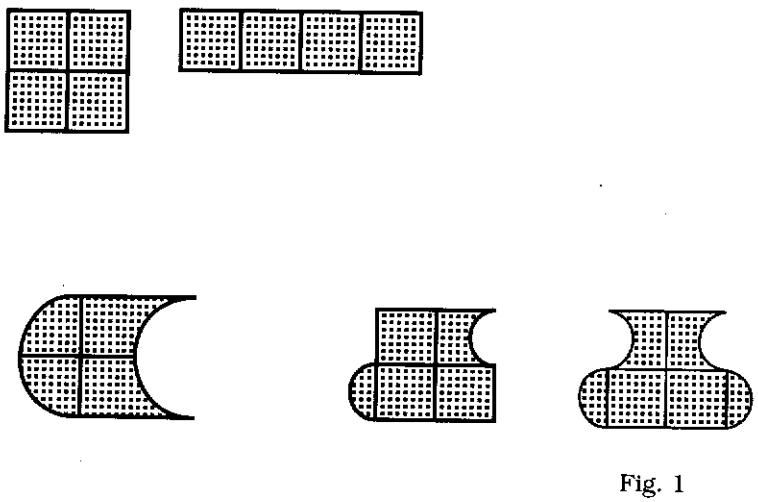


Fig. 1

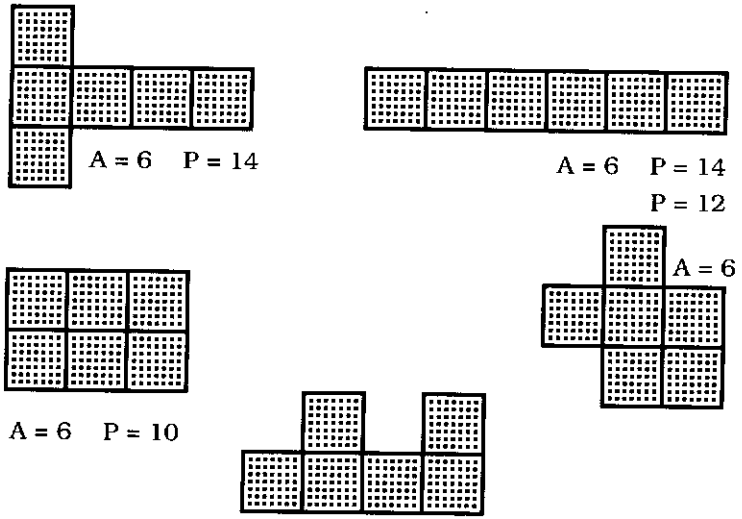


Fig. 2

A= Area P=Perimeter

B 6 – Fun with Cardboard Shapes

1. Prologue

Parents interested in seeing children at creative work on their own would do well to leave them with sand or clay. What excites the children is the instant making and instant changing of shapes. It is in their nature from infancy, without inducement or tutoring from adults, to perform spatial experiments, guided by intuition which provides the initial start in any creative endeavour.

Once they start school, they are prematurely confronted with formal learning to suit the convenience of adults who by and large, have ceased to bother about creativity. Further the survival-urge in children makes them seek the comfort of dependent learning and reject the risks and challenges of relying on their intuitive strength.

It is necessary, therefore, to mitigate the burden of formalised learning by providing the students with opportunities for exercising their intuition, particularly spatial intuition in which they revel. As present day research bears out that the growth of mathematical understanding in children is facilitated if they are encouraged to exercise their spatial intuition.

Composing shapes with pieces gives them the initial start. What is needed is a square cardboard cut into a finite number of bits. Ask children to rebuild the square with the bits. Similar rebuilding puzzles can be set-up with other shapes. Recomposing two squares into bits so as to build a single square is an interesting puzzle that forms the basis of the Pythagorean property of right triangle.

Composing different shapes with the same number of pieces is the next stage and this is the basis of the famous Chinese game of Tangram. The underlying mathematical principle is the conservation of area.

More exciting and more rewarding is the game of dissecting a shape into bits so as to recompose the bits to form a different

shape as required with, of course, no change in area. This provides for the children of primary school age, an element of manageable challenge and extensive scope for enquiry and exploration, in acquiring motivated experience in spatial relations.

2. Regular hexagon into a parallelogram

Children need to get started until they have developed a high degree of involvement. Give them preferably a thin cardboard regular hexagon (sides equal and angles equal). Show them the shape of a parallelogram. Ask them to cut the cardboard hexagon into suitable bits so that they could build a parallelogram with the bits; that is to say in other words, to convert the hexagonal shape into a parallelogram shape. The step by step picture strip explains the technique (see Fig. 1).

3. Quadrilateral into a parallelogram

Next give them a cardboard in the shape of a quadrilateral (four-sided figure). Ask them to cut it suitably into bits to make a parallelogram. The dissection of hexagon does not involve the use of midpoints of the sides. But in the case of conversion of very many rectilinear (bounded by line segments) shapes, *the magical role played by midpoints of sides, presents itself.* Study the picture strip given in Fig. 2.

Ask children to convert through dissection and recomposition any kind of quadrilateral such as a parallelogram, rhombus, trapezium and kite into a rectangle and preserve their work in an album.

Once children discover one way of *doing a certain thing, they seek different ways of doing the same thing.* This gives them a sense of achievement and genuine pleasure.

4. A Quadrilateral into a triangle in two ways

Converting a quadrilateral by *dissection and recomposition process* into a triangle without changing the area is very interesting. There are often more than one way of doing the conversion, an example for which is provided in figures 3 and 4. Children can be seen to develop an insight into the magical role played by midpoints of sides.

Midpoints need not always be taken only on sides. Sometimes midpoints of lines joining the midpoints are also helpful as seen in Fig. 5

5. Square into a triangle

Another such interesting case arises in changing a square into a (scalene acute angled) triangle by dissection as pictured in Fig. 5.

6. Project

Set some easy conversions as project work, e.g. conversion of a rectangle into an isosceles triangle, or right triangle by dissection. A mathematically gifted child can be easily spotted out, if projects of this kind are given as annual contests in schools. Solids also lend themselves to shape conversion puzzles of wonderful complexity.

7. History

Bolyai (1802-60) a great Hungarian mathematician, who is remembered for his contributions to non-Euclidean geometry was the first to pose the *general problem* of dissecting a given polygon of any number of sides into a finite number of *polygonal pieces, so as to recompose them to form another polygon of the same area.* It was left to Hilbert (1862-1943) the great universalist mathematician of this century, to demonstrate the possibility of this general conversion.

Fun with Cardboard Shapes-I

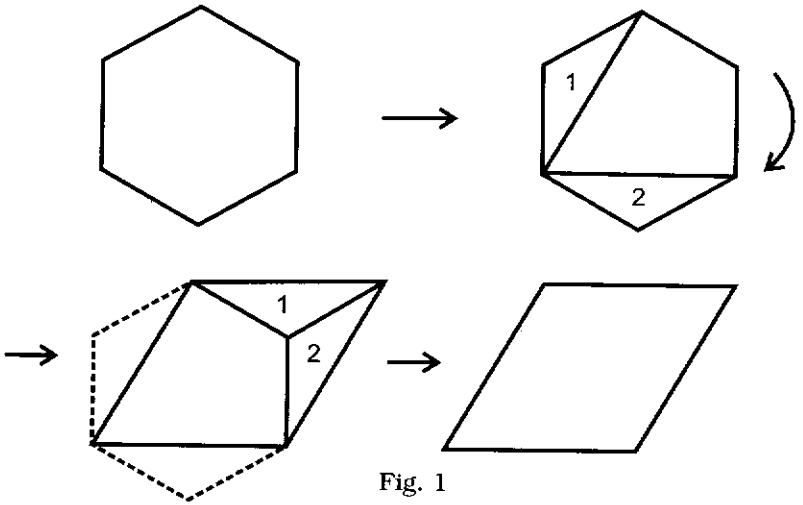


Fig. 1

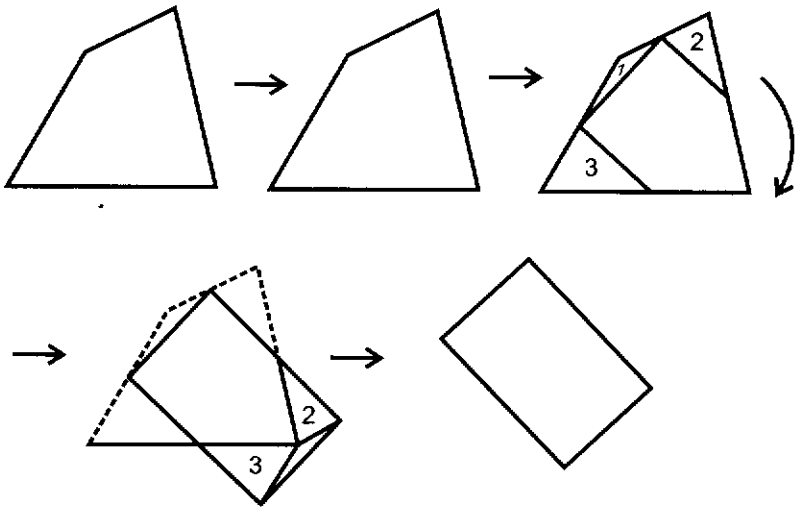


Fig. 2

Fun with Cardboard Shapes-II

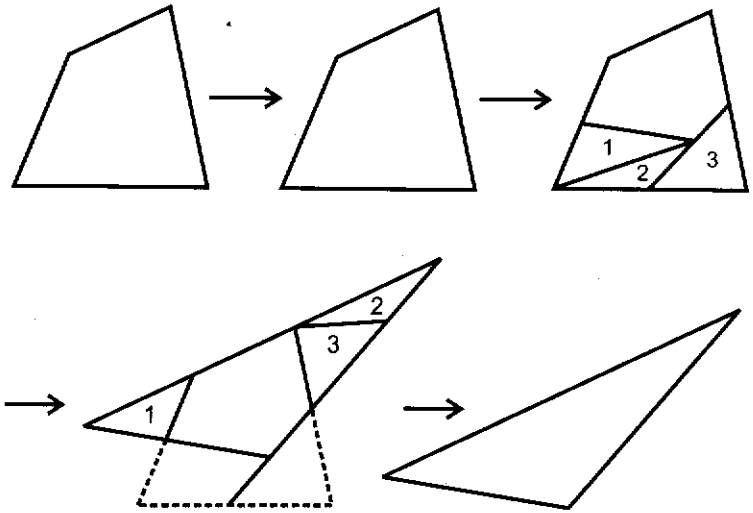


Fig. 3

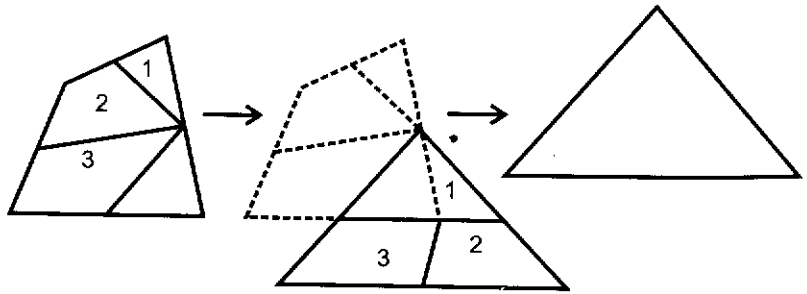


Fig. 4

Fun with Cardboard Shapes-III

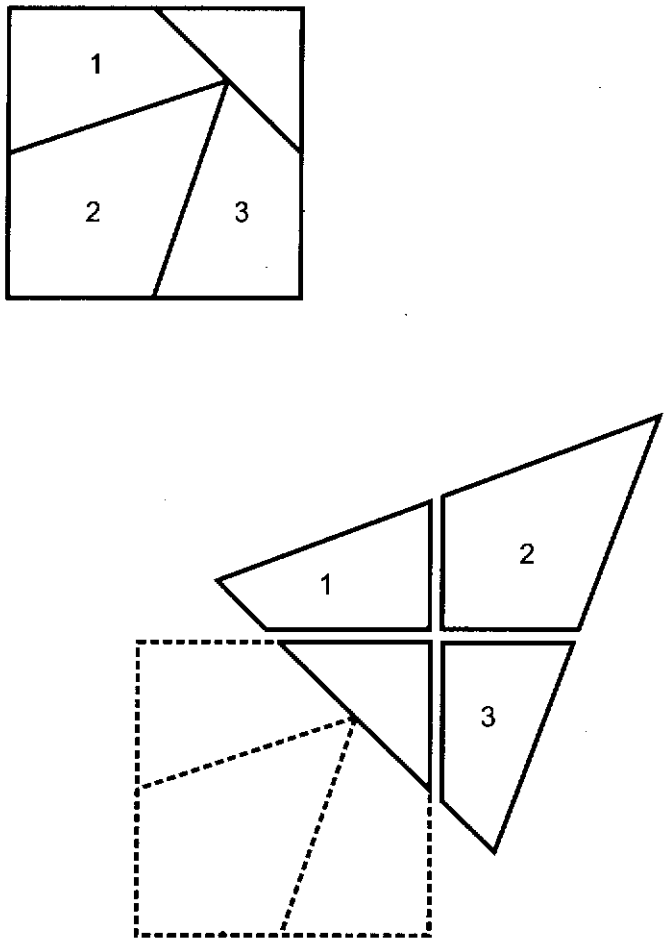


Fig. 5

B 7 – The Nets that Make Up a Box

1. Prologue

It is said that familiarity breeds contempt. If it be so, more serious is complacency created by familiarity. One of the great objectives of education is to develop in children a questioning mind that does not assume that there is nothing more to know about anything just because it is familiar or commonplace. Mathematics affords opportunities to develop this desirable attitude.

Children, particularly those living in urban areas, have no difficulty in collecting cardboard boxes. A drug store can be approached for discarded empty cardboard boxes. Ask children to collect cubical boxes. Many of them know how to make an open cardboard box, using five cardboard squares of equal size.

2. Nets and open cubical box

The question that can provoke thinking through experimentation or otherwise is to find out whether there could be nets with different orientations to make the open cubical box. If that be so, how many nets can be counted distinctly, after discarding those that can be secured through reflection or rotation. Even for adults who face this question for the first time, it is not easy to answer, unless they are endowed with extraordinary spatial intuition.

There are two approaches for solving this problem. One is to take a number of cubical cardboard boxes and flattening each of them by cutting along some edges. Avoid cutting along all the edges, otherwise five separate squares will be obtained. The choice of edges to be cut along will determine the net. The number of distinct nets that are possible can be thus obtained. The other approach is to take sets of five cardboard squares of the same size, make all possible configurations with them, each of which has edge-to-edge complete alignment without jutting, paste slips or put cello-tape across the joined

edges and test by folding upwards if an open box could be got from each. *A configuration of square pieces becomes a net of an open box if it can be folded up to make an open box.*

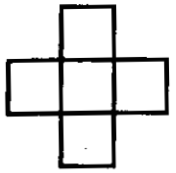
3. Enumeration

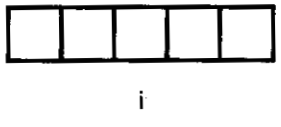
There are 12 *configurations and eight of them are nets.* The twelve configurations are pictured below. If children have finished experimentation, ask them to spot out the eight nets.

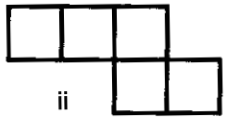
4. Project

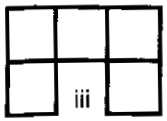
This is an excellent group activity for children and does not require any computational work. Children find it interesting to point out in each net the square that would form the bottom of the box, before the net is folded up to make a box. By fixing a sixth square of the same size, to go with a suitable square of a net, a closed box can be made.

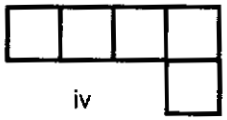
Nets that Make up a Box

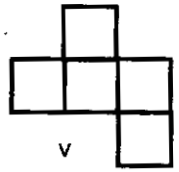


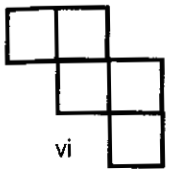


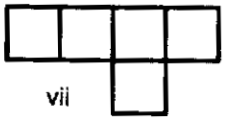


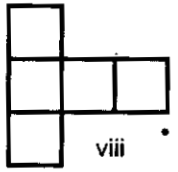


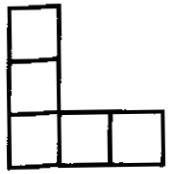


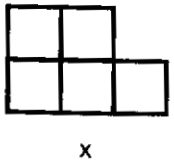


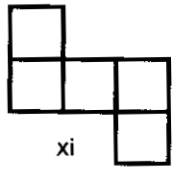


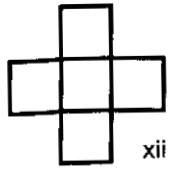












B 8 – Shapes with Set Squares

1. Prologue

Every school-going child is required to possess and carry a box of geometrical instruments. It is a pity that some of the instruments in the box do not receive enough attention and so are not put to adequate use, thereby depriving children of the fine opportunity to know more geometry through experience gained by exploration and experiment, besides what is gained through more conducted practical work.

Every box of geometrical instruments has in general a *graduated ruler* (or a 'scale' in Indian parlance), *a pair of compasses*, *a pair of dividers*, *a protractor* and *a pair of set squares*, one of which is a 30° (or 60°) and the other a 45° one.

2. Project

Versatility of set squares in providing learning experiences is so high that there will be universal welcome if only it becomes better known and appreciated. An interesting project involving the set squares can be assigned to middle-school upper primary children. Ask a child to collect some other set squares, their sizes remaining the same from his/her friends and arrange, first of all, pairs of them in juxtaposition. This helps them to identify the set squares in a more natural and effective manner. 45° set square is half of a square and 30° half of an equilateral triangle (see Fig. 1).

This preliminary exercise sets the stage for the project of building shapes by using several pieces of each set square and arranging them in juxtaposition. If need be, paper models can also be used and pasted to display the combination figures. Some of the shapes built by children untutored with 45° set squares and 30° set squares are shown below (see Fig. 2).

Incidentally children can be introduced to naming of shapes and their parts and, spelling out their properties by

observation through comparison of measures of their sides, angles and diagonals.

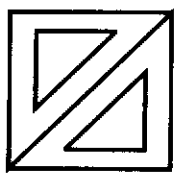
Four-sided shapes which have opposite sides parallel are parallelograms, having their opposite sides and opposite angles equal. Rectangles are also parallelograms with all angles being right angles. Some of the three sided shapes formed are right angle triangles, having one of the angles a right angle (or one of the corners a square corner). The longest side (hypotenuse) of a right triangle is seen to be twice the segment whose ends are the right angled corner (vertex) and the mid point of the longest side. The smallest side in 30° set square that is the side opposite to 30° angle is half of its longest side.

In a four-sided figure with all the four sides equal (square and rhombus), the two diagonals bisect each other and cut at right angles. In the four-sided figure formed with four 60° set squares, one pair of opposite sides are equal and they are slant sides. The other pair are parallel and the figure is a trapezium and so on.

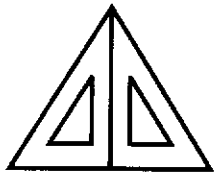
3. Properties in General

These are properties of *particular* figures and children can be encouraged to examine if such properties exist in more cases to know if they are of a general character. Children who thus acquire a rich fund of experience can be seen appreciating refinement and in going about with confidence and care in identifying and recalling with little confusion or hesitation, properties of some of the frequently met figures in school mathematics: rectangle, square, parallelogram, rhombus and trapezium.

Shapes with Set-Squares

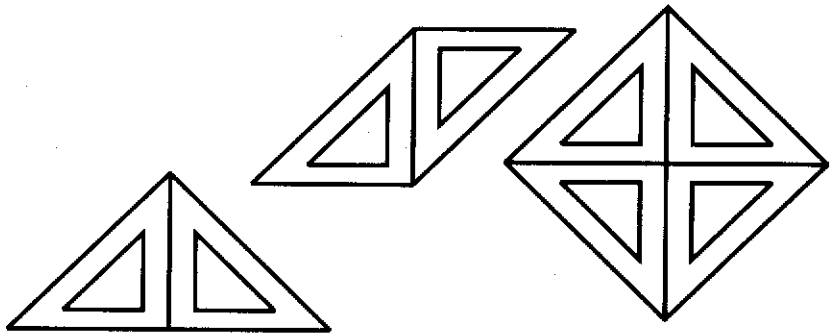


45° Set Squares half of a Square



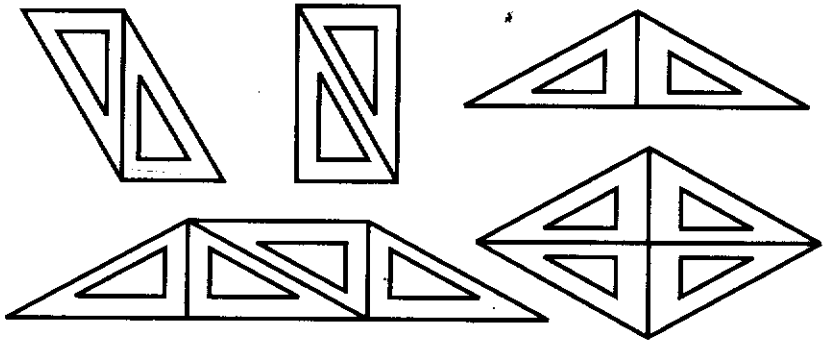
30° Set Squares half of an Equilateral Triangle

Fig. 1



Combination Figures with 45° Set Squares

Fig. 2



Combination Figures with 30° Set Squares

Fig. 3

B 9 – Instant Construction of Solid Shapes

1. Prologue

Transforming shapes is a source of joy. When the process does not involve tearing, cutting or pasting, it becomes ideal work for children, as they find it inviting. Even a child of ordinary ability feels confident to take up such creative and exciting work.

In these days of information explosion, no home is short of periodicals and magazines. What is required for this simple project is a set of four issues of the same magazine with about 100 pages each.

2. Project

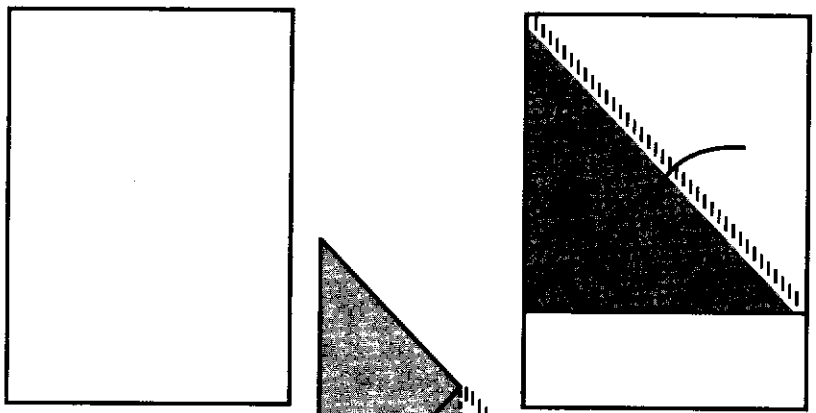
Ask children to fold each sheet as shown in Figs. 1 to 3. All the foldings are towards the binding edge and the binding remains intact.

After all the sheets in the four magazines are folded in the same way, ask the children to set up the folded issues in such a way that they meet at the binding edge, with the binding edge kept vertical on a flat surface (see Fig. 4).

To their pleasant surprise a solid compound is formed with cylindrical shape at the bottom and conical shape at the top (Fig. 5). Ask children to identify the radius of the cylinder as well as the cone and their height. This experience can be recalled, when volume and surface area through integral calculus are taken up for study in higher classes.

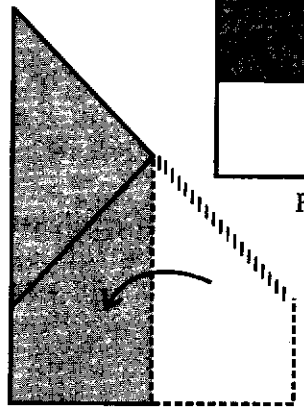
3. More projects

Ask children to think of modifications in folding the sheets so as to get (i) a cylinder capped with cone on either side and (ii) a double cone. Leave children to make their own innovations and find the outcome.

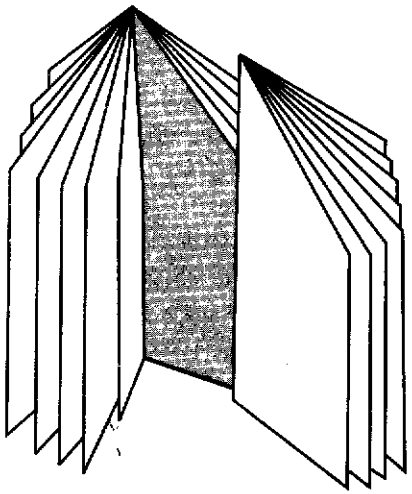


A Sample Sheet
Fig. 1

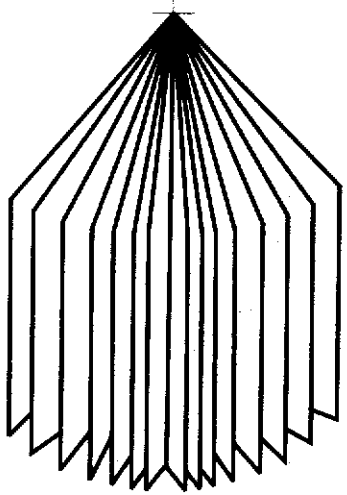
Folding Step
Fig. 2



Folding Step 2
Fig. 3



Four Segments
brought together
Fig. 4



Final outcome
Fig. 5

C. Fun with Numbers

C 1 – Fun Time with Calendars* - I

1. Prologue

On every new year eve, one observes businessmen pleasing their customers and friends with gifts of calendars and all classes of people hunting for as many calendars as possible. Though calendars are mainly used for the purpose of knowing the month, date and day, a variety of them are sought for adorning walls in homes and hotels, shops and offices, serving simultaneously as media of advertisement.

Anyone caring to inquire about the arrangement of numbers on a calendar sheet would be pleasantly surprised to discover numerous wonders. If only parents familiarise themselves with the wonders, they can give their children hours of new discovery's and enjoyment.

2. Patterns

Consider, for example, April 1991 calendar sheet or for that matter the sheet of any month of any year past, present or future.

Sun		7	14	21	28
Mon	1	8	15	22	29
Tue	2	9	16	23	30
Wed	3	10	17	24	
Thu	4	11	18	25	
Fri	5	12	19	26	
Sat	6	13	20	27	

The first thing that one notices is the arrangement of numbers in seven rows in a calendar sheet (or in columns in some calendars). Taking the arrangement in the specimen given above, we see arithmetic progressions or additive sequences.

	Example	Common difference
Row wise sequences	1, 8, 15, 22, 29,	7
Column wise sequences	14, 15, 16, 17, 18, 19, 20,	1
Diagonal wise (left top to right bottom) sequence	1, 9, 17, 25	8
Diagonal wise (left bottom to right top) sequences	5, 11, 17, 23, 29	6

Now take any of the above sequences of numbers. *It is easy to give the mean of any of them, as they are all in arithmetic progression and hence equally easy to give the sum of numbers in any of them.* If the number of terms in an additive sequence is odd, the mean of the numbers in the sequence can be discovered to be the middle term of the sequence. For instance, the mean of the numbers of the sequence.

1, 8, 15, 22 and 29 is 15

and the mean of the numbers of the sequence.

5, 11, 17, 23 and 28 is 17.

But if the number of terms in an additive sequence is even, the mean can be discovered to be half the sum of the end terms or equally placed terms in the sequence.

For instance, the mean of the numbers

1, 9, 17, 25 is

(1 + 25)/2 or (9 + 17)/2, that is, 13

It can be discovered that *half the sum of end numbers or equally placed numbers is a general way of finding the mean and hence applicable to any additive sequence, irrespective of the number of its terms being odd or even.*

Once the mean of number in an additive sequence of finite number of terms is known, the sum of numbers in the sequence is easily obtained by multiplying the mean by the number of terms in the sequence.

Parents can pose the problem of finding by inspection the mean and sum of any additive sequence on a calendar sheet

by demonstrating some examples and asking for justification of the method in general terms or proof. The proof consists of considering arithmetic progressions such as

$a - 2d, a - d, a, a + d, a + 2d$

and $a - 3d, a - d, a + d, a + 3d$

3. Incomplete magic squares

A complete magic square has its row totals, column totals as well as its diagonal totals the same for example, consider the magic squares of 3rd and 4th orders and check the row totals, the column totals and the diagonal totals, otherwise called the magic sums.

8	1	6		16	2	3	13
3	5	7		5	11	10	8
4	9	2		9	7	6	12
				4	14	15	1

So these are examples of complete magic squares. A magic square falling short of magic sums in three ways is an incomplete magic square. In a calendar sheet one can discover incomplete magic squares of orders 2, 3 and 4. Consider first some 2×2 arrays such as

7	14		17	24		22	29
		or			or		
8	15		18	25		23	30

Find row, column and diagonal totals. What do you find? *Only the diagonal totals are equal.*

Now let us consider some 3 × 3 arrays such as

3	10	17		14	21	28		8	15	22
4	11	18	or	15	22	29	or	9	16	23
5	12	19		16	23	30		10	17	24

We find that as in 2 × 2 arrays, the diagonal totals are equal and in addition, *each diagonal total is equal to the middle row total and the middle column total,* or in short, equal to each of the middle line (rows or columns) totals. So 2 × 2 arrays and 3 × 3 arrays on a calendar sheet are incomplete

magic squares. Another interesting situation is seen to emerge in a 3 × 3 array. *The sum of numbers in four corners is equal to the sum of middle numbers in end rows and end columns or in short in end lines, but the sum is different from the sum of numbers diagonal wise or middleline wise.*

Next, let us consider 4 × 4 array of numbers:

1	8	15	22		3	10	17	24
2	9	16	23	or	4	11	18	25
3	10	17	24		5	12	19	26
4	11	18	25		6	13	20	27

Again as before, the diagonal totals are equal. What about the sum of end numbers in both the middle rows and the sum of end numbers in both the middle columns? The sums are the same as the diagonal totals,
e.g. in the first 4 × 4 array

$$2 + 23 + 3 + 24 = 52$$
$$8 + 15 + 11 + 18 = 52$$
$$1 + 9 + 17 + 25 = 52$$
$$4 + 10 + 16 + 22 = 52$$

Now consider the sum of the corner numbers in the 2 × 2 square array formed in the middle. The sums are discovered to be the same as before. Also, if we consider the sums of numbers in opposite corners of a 4 × 4 array and of a 2 × 2 array in the middle of it, we discover the sums to be the same but different from the diagonal totals. This property is discovered to exist in a 3 × 3 array but in the case of a 2 × 2 array, it degenerates into equality of diagonal sums.

By extending the arrangement of numbers on a calendar sheet up to and beyond 31, one can study other higher order square arrays and discover some general properties of incomplete magic squares. It is interesting to build incomplete magic squares in general terms with different characteristics in magic sums.

4. Cross puzzles

The discovery of incomplete magic squares in a calendar sheet leads to the construction and solution of cross puzzles. Incomplete magic squares of odd order form the basis for cross puzzles. Consider the cross of order 3 with 5 spaces arranged horizontally and vertically as in Fig. 1

5. The puzzle

The puzzle is to fill up the five spaces with numbers in such a way that the horizontal sum is equal to the vertical sum. The numbers in the middle column and the middle row in any 3 × 3 incomplete magic square gives the solution. The puzzle becomes interesting if a specific number is given to be the horizontal or vertical sum. What are the steps? How to solve the puzzle? The first step is to fix the number in the central space (junction space of horizontal and vertical lines). Obviously it has to be a third of the given number. Once it is discovered *find two different additive sequences of three terms with the middle term equal to a third of the sum number.* In general, given the sum number a, the central number is $a/3$ and the solution to the cross puzzle can be exhibited as in Fig. 2

Where d and e can be chosen to be non-zero unequal fractional numbers (inclusive of whole numbers). It will be preferable for d and e to be less than $a/3$ so as to avoid negative fractional numbers in any of the boxes.

* Later developed into an enrichment book *in dialogue* mode for children under the caption Number Fun With A Calendar, written and published by the author in 1988.

Fun Time with Calendars-I

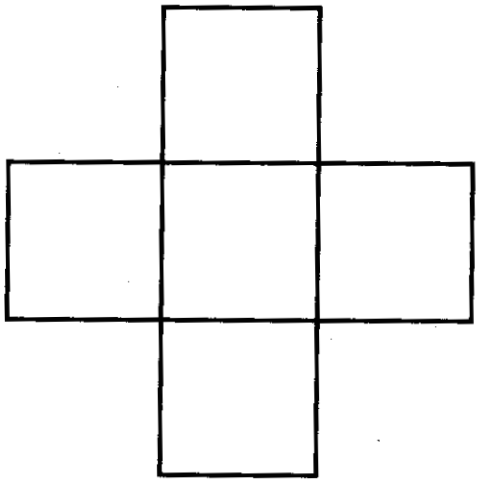


Fig. 1

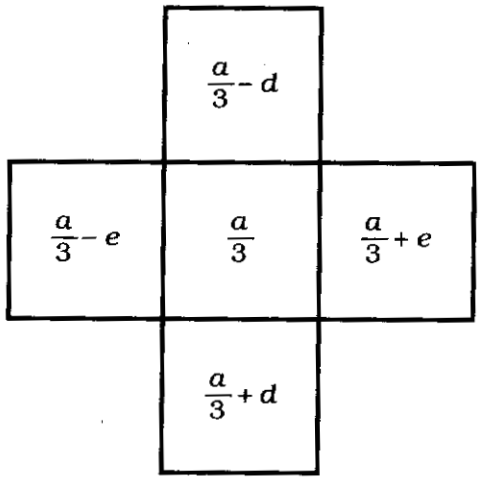


Fig. 2

C 2 – Fun Time with Calendars - II

1. Star Puzzles

Incomplete magic squares of any order form the basis for construction and solution of star puzzles. Consider the star puzzle of order 2. The puzzle is to fix numbers in four spaces in such a way that diagonalwise or diameterwise pairs have the same sum (see Fig. 1).

Any 2 × 2 incomplete magic square gives the solution. If any particular sum of an opposite (diagonalwise or diameterwise) pairs is required and given, the puzzle becomes more interesting. An easy way to the solution of the puzzle is by using half the sum to fix numbers in opposite places diameterwise. The figure $a/2$ itself will not find a place. In general terms, the solution of a star puzzle of order 2 is exhibited as shown in Fig 2e and d can be chosen to be non-zero unequal fractional numbers (inclusive of whole numbers) less than $a/2$.

This will help you construct and solve star puzzles of orders 3 and 4 or of any order for that matter (see Figs. 3 and 4).

2. Location Game

Take any two or more rows. Choose a number (consider the calendar sheet given in C1) at random in each row. Find the sum of the numbers chosen. See that the sum does not exceed 30°. The game is to locate the row in the calendar sheet in which the sum would occur, without knowing the numbers chosen. In locating the row in which the sum occurs, without knowing the numbers chosen, one discovers the key role played by the leading numbers (or the smallest numbers) with which the two rows commence. *The row which commences with the sum of the leading numbers of the rows chosen, becomes the sum row.*

For example, take the third row commencing with 2 and the fifth row commencing with 4 in the arrangement of numbers in the specimen calendar sheet given earlier. Let

the numbers chosen from the rows be 9 and 18 respectively. Their sum is 27 and it is seen to occur in the seventh row commencing with 6 which is nothing but the sum of the leading numbers 2 and 4 of the rows chosen. If the sum exceeds 30, it cannot be seen in the sheet. One has only to consider in that case number arrangement in a calendar sheet extended up to and beyond 31 to locate the sum in the sum row. *This is the sum row location game.* With this experience one can easily construct and play *product row location games.* Even *difference row location game* can also be constructed and played.

3. Modular Arithmetic

The row location game described above is simply an interesting application of modular arithmetic for module 7 or in other words arithmetic of remainders or residues obtainable on division of whole numbers by 7. A little study reveals that the numbers in each row leave the same remainder as the leading number with which the row commences. For instance, take the sixth row of numbers 5, 12, 19, 26 each of which when divided by 7 leaves the same remainder 5, and 5 is the leading number of the row.

In standard mathematical language, 5, 12, 19 and 26 are considered to be congruent for module 7 and the mathematical statements are as follows:

$5 = 12 \pmod 7$

$12 = 26 \pmod 7$ and so on

The relation is called congruence and it has the same properties as equality relation in respect of addition, subtraction and multiplication. The behaviour of congruence relation needs to be watched in respect of division as it is somewhat different from that of equality relation, depending on the nature of module being prime or composite. One way of finding whether a given statement involving this congruence relation is true is by checking if the difference of numbers on either side of congruence relation is a multiple of 7.

For instance, given $12 = 26 \pmod 7$ we can see that the following derived statements are also true:

$12 + 3 = 26 + 3 \pmod 7$

$12 \times 3 = 26 \times 3 \pmod 7$

Many congruent statements can be derived by adding (or subtracting) a multiple of the module to (or from) either side of the congruence relation e.g. $12=26+2x7 \pmod 7$.

4. Naming the day

You are given a date with its corresponding day and you are asked to name the day for a date after or before a certain number of days from the given date. How will you find it ? Of course one can do it by counting dates (or skip counting by weeks and counting the remaining dates) forward or backwards on a calendar, extended if necessary both ways and reading the corresponding day. But a little study of the calendar suggests better, more interesting and quicker methods of solving the problem.

Refer to the specimen calendar sheet cited at the beginning and examine if the following statements are true.

10 days after 9th date is of the same week day as 3 days after 9th date or 3 days after 2nd date (Note: $10 = 3$ and $9 = 2$ for module 7). Twelve days after 17th date is of the same week day as 5 days after 17th date or 5 days after 3rd date. (Note: $12 = 5$ and $17 = 3$ for module 7).

Try to make many more such statements with reference to the specimen calendar or any calendar sheet for that matter.

In the above statements *replace* after by *before* and see if the resulting statements are true. These are nothing but interpretation of statements involving congruence relation for module 7 in the context of arrangement of numbers associated with week days on a calendar sheet.

The truth of these statements draws our attention to the importance of the leading numbers 0 to 6 which are the residues in module 7 system (Note: $0 = 7 \bmod 7$). This enables one to discover the short cuts in finding the day for a date in the past or future, once a date with its corresponding day, is known.

Suppose you are asked to find the day 389 days after 23rd June '80. What is the short cut? Transform the problem to that involving their residues (that is, remainders got on

division by 7) viz. 4 and 2. *Getting the residues for a number is done easily by removing as many multiples of 7 as is possible from the given number.* For example:

$380 - 50 \times 7 = 39$

$39 - 5 \times 7 = 4$

So $389 = 4 \pmod 7$

$23 - 3 \times 7 = 2$

so $23 = 2 \pmod 7$

In such a manner the problem gets reduced to finding the day 4 days after 2nd that is, the day for 6th date which is Saturday. So 389 days after 23rd will be a Saturday.

Thus a calendar gives a beautiful setting for getting familiar with module 7 arithmetic. By modifying the calendar to suit a 6 day week working calendar, one could get introduced to module arithmetic in general which was given a definitive development in the 18th century, by Gauss, one of the most creative mathematicians of great renown.

Versatility

The calendar holds many more mathematical treasures such as integers (through extension of dates both ways of a calendar for a month), algebraic structures, mappings and their preservation. In other words, the calendar abounds in wonderful models of mathematical systems.

Fun Time with Calendars-II

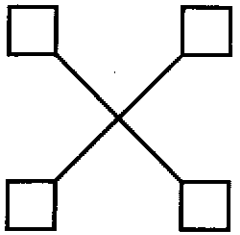


Fig. 1

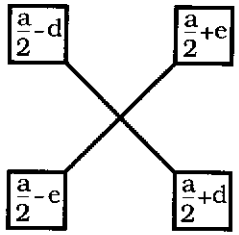


Fig. 2

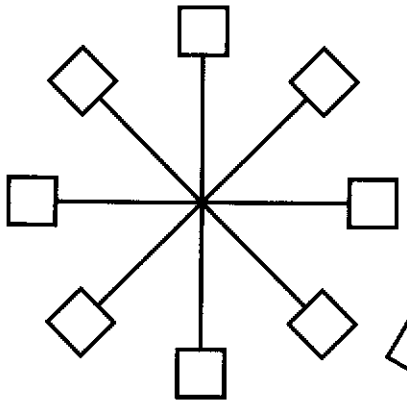


Fig. 3

The puzzle of order 3 or 8 spaces

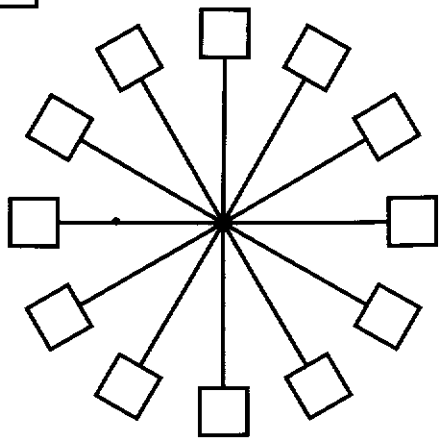


Fig. 4

The puzzle of order 4 or 12 spaces

C 3 – Magic Squares for Greetings

1. Prologue

Magic squares always cast a spell on the young and the old, mathematicians or non-mathematicians. Interest in magic squares was evinced by ancient Chinese and Indians. Narayana Pandita of the 14th century has dealt extensively with magic squares in his Ganita Kaumudi. The first chapter in Ramanujam's notebooks is on magic squares. Euler (1707-83), the most prodigious mathematician known to the world devoted a lot of his attention to magic squares.

What is a magic square? If in a square matrix of different numbers, the row totals, the column totals and the diagonal totals are the same, the square matrix becomes a magic square. The common total is called the magic sum. Apart from getting the thrill, a positive attitude is also created in children when they manage to construct a magic square.

Children are usually shown some magic squares which they remember and repeat. Sometimes they learn to build them with any set of consecutive numbers having a square number count. But to make the square construction more interesting, birthday greetings, which mean a lot to people provide the opportunity. Building a birthday magic square and making it a part of greetings will certainly invest it with special attraction.

A birthday as a date consists of four numbers relating to the entries for the day, the month, the century and the year in it. With these four numbers taken in their order in the four consecutive cells of the first row (or column), the magic square is built. Such a magic square naturally consists of 4 × 4 or 16 cells (hence called a fourth order magic square). The numbers that go into the cells should all be different. So if a birthday, or any date for that matter has any two of the same, entries it will not be magic square worthy.

As is the case with magic squares, there are different ways of building a fourth order magic square. Moreover there is no unique single solution. Since it is intended for the enjoyment

of children, once they have developed confidence in their skills of addition and subtraction, an arithmetic technique based on symmetry would be appropriate.

2. Sriramachakram (Discovery of patterns)

Clues for the technique are discovered from a study of Sriramachakram given in Hindu almanacs. Sriramachakram is traditionally used for instant, astrological predictions of a sort but our interest here is in its patterns of partial sums.

Sriramachakram is a fourth order magic square with its cells filled with numbers 1 to 16 to yield the magic sum 34. Before we look for patterns that suggest the building technique, children should be introduced to the identification of corresponding rows, columns or diagonals and end cells and middle cells in any row, column or diagonal.

Among the rows or columns, the first corresponds with the fourth and the second corresponds with the third. The two diagonals obviously correspond with one another. Each row, column or diagonal has two end cells and two middle cells.

The sum of numbers in a pair of end cells is called end-sum and the sum of numbers in a pair of middle cells the mid-sum.

3. Strategy

Figure 1 gives the Sriramachakram magic square. Ask children to spot out the equal sums in corresponding rows, columns and diagonals. Cases of equal sums are easily discovered and an examination of these suggests the construction rule:

1st and 4th	9 + 4 = 3 + 10;
rows:	16 + 5 = 6 + 15;
2nd and 3rd	7+ 14 = 13 + 8;
rows:	2 + 11 = 12 + 1;
1st and 4th	9 + 6 = 14 + 1;
columns:	7 + 12 = 4 + 15;
2nd and 3rd	16 + 3 = 11 + 8;
columns:	2 + 13 = 5 + 10;
two diagonals:	9: 13 = 13 + 11;
	2 + 8 = 6 + 4;

Children discover, to their great delight, *that the end sum of any row, column or diagonal is equal to the mid sum of the corresponding row, column or diagonal and vice versa.* This may be called the *mid-end rule of symmetry.*

This discovery suggests the steps in building a birthday magic square. The process is in two stages and each stage involves composing a two number sum and recomposing the sum obtained into two other numbers, one of which is often seen to be fixed. Care is of course taken to see that no number is repeated anywhere in the cells.

4. Birthday magic square

Ramanujam was born on 22-12-1887. Let us build his birthday magic square. The first row would consist of 22, 12, 18 and 87. The mid sum of this row is 12 + 18 = 30. Decompose it into two other numbers, say 28 and 2, to fill the end cells of the corresponding forth row (see step 1 of Fig. 2). End numbers of the two diagonals are now available. *The end sum of one diagonal should be equal to the mid sum of the other diagonal.* So compose the end sum of a diagonal and decompose it into two other numbers to go into the middle cells of the other diagonal (see steps 2 and 3 of Fig. 2). Now the middle cells of the fourth row can be easily filled by means of the mid-end rule of symmetry (steps 4 and 5 of Fig. 2). With these fillings, the first stage is over. Let children note that the four corner cells and the four middle cells are called VIP cells as each is to be considered in three totals. Start the second stage, fix some convenient number in anyone of the four end cells seen blank. In the figure here, the end cell where the 2nd row and the 4th column meet is taken and filled with, say 11. The other remaining cells are easily filled by means of the rule (see steps 7, 8 and 9 in Fig. 2). Of course, as observed earlier, no two children will get the same magic square. Let the children check and see that the row totals, the column totals and the diagonal totals yield the same magic sum 139.

5. Epilogue

The beauty of this method of construction lies in doing additions and subtractions with only two numbers at a time. The magic sum is automatically obtained. Children develop a feel for large and small numbers while constructing the magic squares.

Magic Squares for Greetings

9	16	5	4
7	2	11	14
12	13	8	1
6	3	10	15

Sriramachakram

Fig. 1

22	12	18	87
28			2

Step 1

22	12	18	87
		9	
	15		
28			2

Step 2

22	12	18	87
	60	9	
	15	55	
28			2

Step 3

22	12	18	87
	60	9	
	15	55	
28	52		2

Step 4

22	12	18	87
	60	9	
	15	55	
28	52	57	2

Step 5

22	12	18	87
	60	9	11
	15	55	
28	52	57	2

Step 6

22	12	18	87
59	60	9	11
	15	55	
28	52	57	2

Step 7

22	12	18	87
59	60	9	11
30	15	55	
28	52	57	2

Step 8

22	12	18	87
59	60	9	11
30	15	55	39
28	52	57	2

Step 9

Fig. 2

C4 – Altering the Magic Square with Minimum Changes

1. Prologue

Whenever an exhibition is held for more than one day in a school, visitors would be attracted by the magic squares built and displayed to mark each of the days. Having children to build a magic square for the opening day by the mid-end rule of symmetry following invocation for the day it would be exciting to build magic squares to mark the subsequent days by *just making a few alterations in the first day's square.* Advertisers can also use magic squares effectively with off-on flashes.

2. Strategy

Suppose a two-day exhibition was held on January 14 and 15, 1984. Let a magic square for 14-1-1984 with the magic sum 118 be as shown (see Fig. 1).

The magic sum for 15-1-1984 the next day's magic square would be one more than 118. So to change the magic square to suit the next day, one has to alter the numbers in the cells in such a way that the magic sum increases by just one. Of course it would not work if 1 is added to each number. If alterations are to be by addition of 1, one number in each row would have to be carefully chosen resulting in the alteration of any four numbers. This would be the least number of alterations. Of course the number in the first cell of the first row would have to be increased by 1.

3. Two ways

The problem takes the form of a puzzle of fixing four is in the cells of the 4×4 square, so as to see that when one looks across any row, any column or any diagonal, 1 appears only once in it. The two possible ways are exhibited below (see Figs. 2 and 3). Each gives the change matrix. In this matrix, 0 (zero) means keeping the number in the corresponding cell

of the original matrix unchanged. Similarly, + 1 means increasing the number in the cell by 1. *On using one of these change matrices if a number repeats itself then the other matrix can be made use of.* The possibility of repetition of cell numbers is however remote.

In case the altered magic square still has repetition of numbers, then the magic square for 14-1-1984 into a magic square for the next day 15-1-1984 by using either of the change matrices. The solutions are as given below (see Figs. 4 and 5).

Children can be asked to suggest change matrices for alterations to be made in a magic square *marking a date to get a magic square for the same day next month or for the same day and month next year.*

Instead of building a magic square with the numbers of a date in the first row itself, they can be assigned the project of finding ways to build magic squares in such cases by means of the mid-end rule of symmetry.

4. Programming

Drawing up a flow chart and a programme for building this kind of magic square would be a challenging project for high school students when exposed to the use of programmable calculators.

Altering the Magic Square

14	1	19	84
58	42	6	12
28	10	60	20
18	65	33	2

Fig. 1

+ 1	0	0	0
0	0	+ 1	0
0	0	0	+ 1
0	+ 1	0	0

Fig. 2

+ 1	0	0	0
0	0	0	+ 1
0	+ 1	0	0
0	0	+ 1	0

Fig. 3

15	1	19	84
58	42	7	12
28	10	60	21
18	66	33	2

Fig. 4

15	1	19	84
58	42	6	13
28	11	60	20
18	65	34	2

Fig.5

C5 – Currency Notes and Fibonacci Numbers

1. Prologue

Situations with constraints are mathematically significant. Mathematical studies of a situation emerge by increasing or reducing the number of constraints governing the situation or by altering the nature of the constraints. Even if a situation has no constraints, constraints can be imposed on it.

Consider the situation of giving any amount in rupees in terms of only *one rupee notes* or *one rupee coins*. The situation does not appeal to a mathematical mind, as it does not create any problem. On the other hand, stipulate that the amounts are to be given in terms only of *two rupee notes and/or one rupee notes*. Not only that, impose a further constraint that *the order of giving the currency notes would be taken into account in counting the number of ways of giving an amount.* Of course, no distinction is made between currency notes of the same denomination.

2. Strategy

In any study, tabulation helps and so let us tabulate the results get as displayed by the figures 1 and 2.

Summarising we get:

Amount in Rupees	Ways of giving the amount in terms of two and one rupee currency notes only with emphasis on the order of giving	Number of ways
0	–	1
1	1	1
2	1, 1 or 2	2
3	1, 1, 1 or 1, 2 or 2,1	3 and so on

3. Birth of Fibonacci Numbers

Ask your child to continue this interesting study, recording the number of ways for each higher amount. Direct the child's

attention to the pattern emerging, if the child has not done so in the meantime. You can see the child discovering with joy the generating pattern showing that any number in the sequence is simply the sum of the two numbers previous to it in the sequence.

1,	1,	2;		2	=	1	+	1
1,	2,	3;		3	=	2	+	1
2,	3,	5;		5	=	3	+	2
3,	5,	8;		8	=	5	+	3

and so on

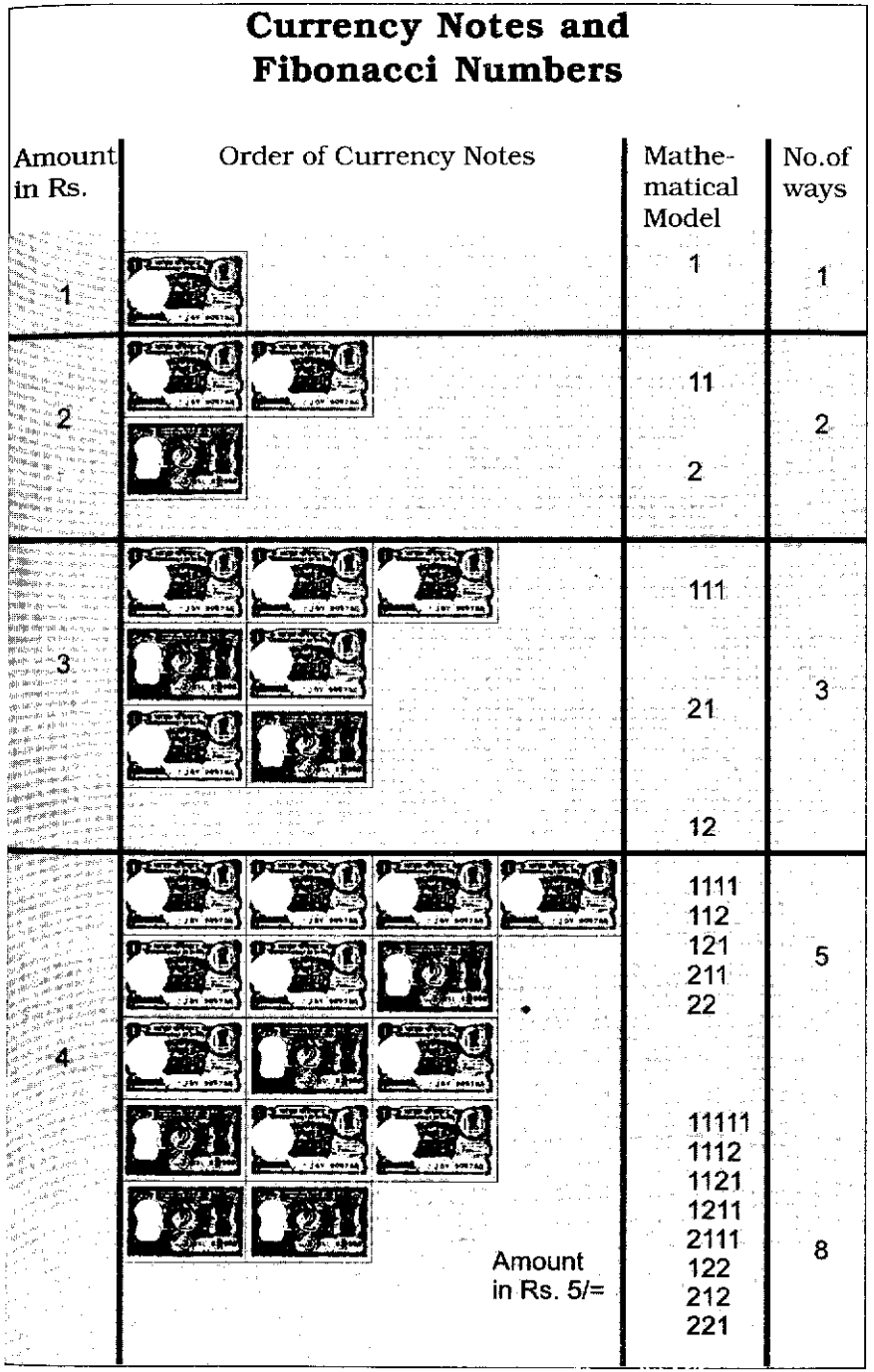
This remarkable sequence is called Fibonacci Numbers, named after Fibonacci, a famous Italian mathematician of the Middle Ages. Fibonacci introduced the Hindu number system to Europe through his influential handbook on arithmetic and algebra, *Liber abaci* (1202) which means Book of abacus. The book aims to show the superiority of Hindu numerals over the clumsy Roman system.

The enchantment of Fibonacci number sequence has not ceased to grow with the passage of time. As a matter of fact, a separate association was started in 1963 in the United States and a quarterly journal has been making its appearance regularly to unravel its beauties which seem to be endless.

Dr. Krishnaswami Alladi, one of our leading young mathematicians has obtained some spectacular results about Fibonacci numbers which have been published in this quarterly journal.

4. The Fascination

The fascination of the sequence lies in its appearance in unrelated and unexpected situations such as seed arrangements on sun flowers, leaf arrangements of plants, spacing of limbs in the human body, poetic compositions, architectural designs, data sorting, generation of random numbers, games and puzzles, etc.



C6 – Some Fascinating Properties of Fibonacci Numbers

1. Prologue

Fibonacci numbers have many fascinating patterns. Some of them can be easily discovered by children who have a natural flair for detecting patterns. The sequence of F-numbers consists of 1, 1, 2, 3, 5, 8, 13, 21, 34, 55 and so on.

2. Some patterns in any pair of consecutive 'F' numbers

The obvious patterns are that the F-numbers except the first one are alternately odd and even, and no two consecutive numbers have common factors other than one. In other words two consecutive F-numbers are co-prime.

3. A pattern in any triad of consecutive 'F' numbers

Taking any three consecutive numbers in the sequence one can see instantly:

1, 1,2	1×2	-1	=	1^2
1, 2, 3	1×3	$+1$	=	2^2
2,3,5	2×5	-1	=	3^2
3, 5, 8	3×8	$+1$	=	5^2
5,8, 13	5×13	-1	=	8^2

and so on.

So in any triad of consecutive numbers the product of the end numbers differs from the square of the middle number by one. You may like to prove it, in general, for any triads of F-numbers.

4. A pattern in any four consecutive 'F' numbers

Taking any four consecutive F-numbers one cannot fail to discover a property underlying them.

1, 1, 2, 3	$2^2 - 1^2$	=	1×3
1, 2 3, 5	$3^2 - 2^2$	=	1×5

2, 3, 5, 8	$5^2 - 3^2$	=	2×8
3, 5, 8, 13	$8^2 - 5^2$	=	3×13

and so on.

That is in any *set of four consecutive F-numbers the difference of squares of the two middle numbers* is *simply the product of the end numbers.* (A proof, in general, should be attempted).

5. Finding sums

Children often raise questions about finding the sums of F-numbers and given the right start, they can discover the answers themselves. There are two simple cases within the understanding of children.

(1)	$1 + 1$	$= 3 - 1$	
	$1 + 1 + 2$	$= 5 - 1$	
	$1 + 1 + 2 + 3$	$= 8 - 1$	
	$1 + 1 + 2 + 3 + 5$	$= 13 - 1$	and so on.

The sum of any number of terms in any sequence starting from the first is simply one less than the term coming in the third place from the last term of the chosen part of the sequence.

(2)	$1+1+2+3+5+8+13+21+34+55$	=	13×11
	$1+2+3+5+8+13+21+34+55+89$	=	21×11
	$2+3+5+8+13+23+34+55+89+144$	=	34×11 and so on.

The sum of any ten consecutive F-numbers is eleven times the seventh F-number.

6. Generalising spree

By raising the question, why not add the previous three terms instead of two terms and thus constructing a new series Tribonacci numbers can be created and their properties studied.

The pattern of F-numbers induces one to try generalising as a result of which another simple sequence was set up by Lucas in the 19th century. It runs thus:

1, 3, 4, 7, 11, 18, ...

F-sequence and its parallel versions could be assigned as an excellent project in combinatorics for the gifted students. Preparation of an album to display F-sequence by using only one paisa and two paisa stamps (used) would be a worthwhile engagement for those interested.

C7 – Number in Kolam

1. Prologue

One of the few pleasing sights that can be witnessed even today as a part of our living tradition is the drawing of kolam every day at sunrise in front of Hindu homes, rich or poor, in rural as well as urban areas, all over the southern states of India. A kolam is essentially a geometrical design made up of lines, straight or curved, against a framework of dots. It is usually drawn by the housewife in the home. It is often watched eagerly by young girls and the girls get so fascinated that as soon as they are at the threshold of puberty, they make bold to engage themselves in this aesthetic achievement as a mark of their growth and maturity.

A kolam can be made as elaborate and complex as one's intuitive strength could take one along. This could be seen on special occasions like festivals, marriages, etc. Anyone keenly interested in enjoying and studying the live shows of kolam cannot afford to miss the month of *marrghazhi* (mid December–mid January) as this month is set apart for kolam extravaganza. The presence of kolam in front of a Hindu dwelling is considered indispensable as absence of kolam is a silent announcement of some sad happening there.

2. A Preliminary Study

On ordinary days, simple kolams of *direct* and *interval* types are usually made. Each kolam is associated with a specific number of dots and this provides a good example to get children interested in the number of dots associated with each kolam. Children do become curious about the arrays of dots but their curiosity is not sufficiently nurtured by elders, as the geometric design of kolam is made to receive greater attention. Every kolam starts with fixing of a certain number of dots, as the dots provide the guidance for drawing of lines with ease for the kolam.

There is a *base line of dots* that determines the size and shape of kolam-framework. In the *direct type* of simple kolam

framework, dots are placed in rows directly below or above the dots in the base line and the number of rows is the same as the number of dots in the base line. The simple direct-kolams are generally made up of curves. In the case of an *interval type of* simple kolam framework, the dots on the top and bottom rows are not placed directly below or above the dots on the base line but in interval spaces in such a way that the number of dots gets progressively reduced until the last row has only one dot. Simple interval-kolams are usually made up of line segments (see Fig. 2).

If, in the interval type, dots are placed above as well as below the base line, *standard* interval type *of* simple kolam is obtained and the arrangement of dots takes a rhomboidal shape.

Children love to make on their slates just the framework of dots and make the first discovery that the number of dots constituting a simple direct-kolam is the same as the number of dots in a standard simple interval kolam. If children have been familiarised to recognise squares and square numbers, triangles and triangular numbers, then they will realise that this number in each case is a square. The number of dots in a simple interval kolam is a triangular one and this background serves children well in making a little deeper discovery that framework of dots in any standard simple kolam, standard interval as well as direct, is built up with two consecutive triangular numbers (see Fig. 3).

3. Some number sums generalisations

Children can involve themselves in simple generalisations at this stage. A child can be helped to state verbally and then symbolically that if the base line of a *simple interval-kolam* has n dots, there will be $(n - 1)$ rows on either side and the single dot appears in the $(n - 1)$ th row. By suggesting to children to fix another *simple interval-kolam* of the same size by the side of a given one but in the reverse order or inverted form, children see dots assuming the shape of a parallelogram. The number of rows in any such parallelogram is one less than the number of dots in a row (see Fig. 4). If the number of rows is n, the number of dots in each row is $(n + 1)$ and the total number of dots in each such parallelogram is $n(n + 1)$.

This understanding enables children to see next that the number of dots in a *simple interval kolam is half the number of dots in the framework of dots of the corresponding standard interval-kolam.* So the number *of* dots in a simple interval-kolam is $1/2\ n\ (n + 1)$, where n is the number of dots in the base line.

4. Further studies

There is rich mathematics in kolam art which finds a place in Graph Theory. Kolams can be studied for their symmetries, topological structures, etc. It is unfortunate that this great cultural resource finds no mention in our mathematics textbooks. Kolam arrays can be studied in depth as has been so ably demonstrated by Dr. Rani Siremoney, former Professor of Mathematics, Madras Christian College, whose contributions in formalising their arrays have received international recognition.

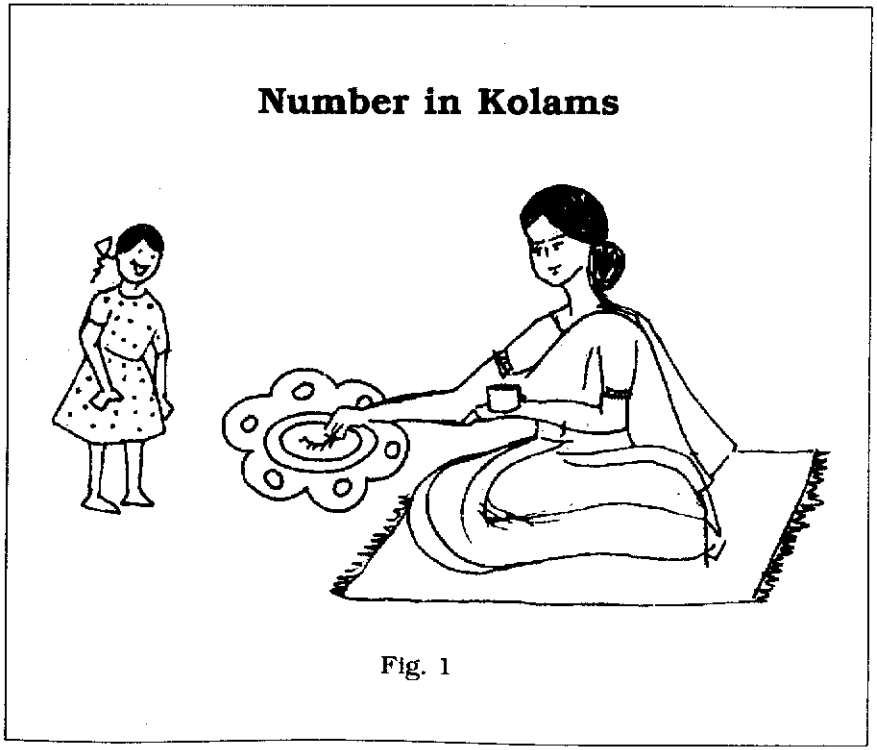


Fig. 1

Simple Kolams

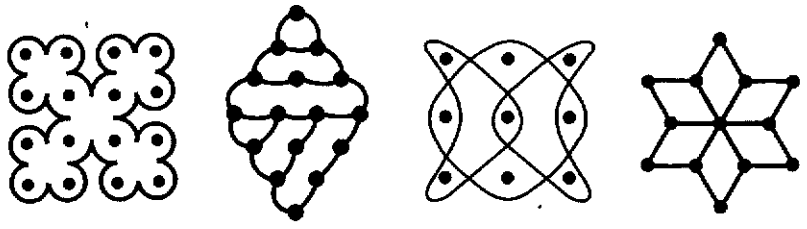


Fig. 2

Simple Kolam Dot Arrays

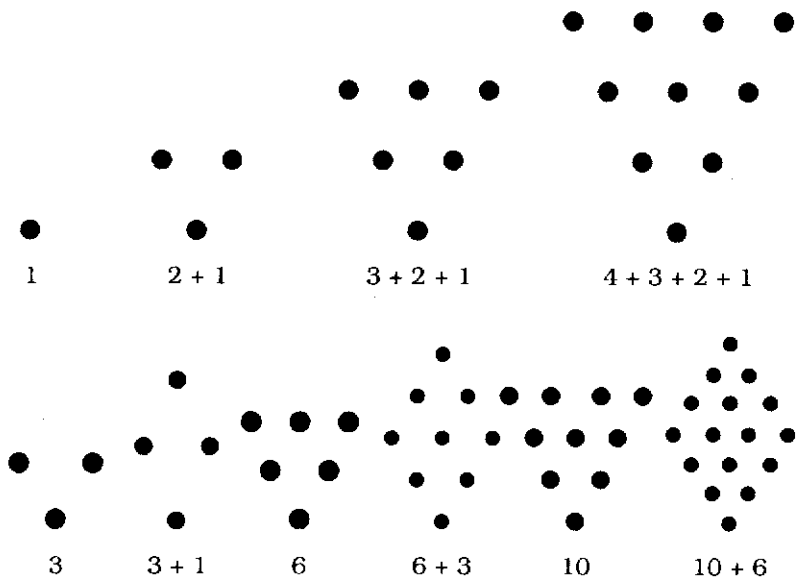


Fig. 3

D. Games for Enrichment

D1 – Bowls Set and Transfer Game

1. Prologue

Almost in every home a set of bowls (or containers) plastic or metal, with such sizes that they can be put one inside the other, starting from the largest, can be found. An interesting game called the Game Transfer can be played in any home, say after supper (see Fig. 1a).

2. Choice

Ensure that rims of all the bowls in the set are at the same level or one inside the other. No rim of any bowl should be seen projecting (see Fig. 1a). Take one such set. Place three circular plates, metal or plastic in such a way that they touch each other as in Fig. 1b. Let each plate be large enough to hold the set of bowls, the plates need not be of equal size.

3. The game

Place the bowls set on one of the plates as shown above in Fig. 1c. The objective of the game is to *transfer the set as it is to either of the two remaining empty plates by observing some rules*. The plates can be used *in any order* while making the moves. The rules of the transfer game are: (1) only one vessel can be moved at a time; (2) the transfer should be made in the least number of moves.

With one vessel the transfer can be made in one move. With a set of two bowls, the transfer can be made in three moves as shown in Fig. 2a.

Let children increase the number of bowls in the set to three and four and play the transfer game observing the rules. Let children count the least number of moves in effecting transfer of the set for cases with one, two, three, four, etc. bowls and tabulate their findings as suggested below:

(The blanks are to be filled by children from their own findings).

No. of bowls	Least no. of moves
1	1
2	3
3	–
4	–

Ask children to study the pattern of numbers in the number of moves column and predict the least number of moves for a set of five bowls and verify by actual play. If necessary, *give the hint that children can study the successors of numbers in the moves column and their relationship to corresponding numbers in the number of bowls column.*

Children discover that the *least number of moves can be given by means of the formula* $2^n - 1$, *where n denotes the number of bowls.*

4. A Deeper Game

The game can be made exciting by specifying one of the remaining two plates for effecting the transfer. Help children observe and make the incidental discovery that the final transfer for *odd* number of bowls is in the plate where the first plate was moved itself, whereas for *even* number of bowls it is in the other plate where the first move is not made.

Now with one bowl, the least number of moves for transfer is 2 (see Fig. 3a).

With two bowls set, the least number of moves for transfer is discovered to be 8 (see Fig. 3b).

As before, let children increase the number of bowls to three and four and make the transfer and count the least number of moves in each transfer and tabulate their findings. Let them study the patterns of numbers in the moves column and discover the formula. The formula can be found to be $3^n - 1$, with the usual meaning for n.

6. Introducing sophistication

By way of sophistication, dolls with their heights in gradation, discs with their diameters in gradation or number cards with number values in gradation can be employed for this game of transfer.

One more rule needs now to be added for this variation. *The rule is not to move a thing of higher order (in height, diameter, or number), to the place occupied by a thing of smaller order (in the corresponding characteristic).* This rule has not found a place earlier, as it is built into the system with the set of bowls of graded sizes, facilitating the making of moves.

The game goes by the name of *Tower of Brahma* or *Tower of Hanoi.* The reason for the name involving Brahma is given in the form of a story. The reason for the other is obscure.

7. The story

In Varanasi, there is supposed to be a great temple and its dome is supposed to point to the centre of the universe. A plate of brass with three diamond needles fixed on it lies below the dome. At the same time of creation, the Almighty was said to have placed sixty-four golden discs on one of the needles. The largest disc was at the bottom, with the others arranged over it in the gradation of their sizes. This device has come to be called the Tower of Brahma.

Priests are reported to have been transferring without a break, discs from one needle to another according to the laws of Brahma which stipulate that (1) only one disc should be moved at a time, (2) no disc should be so placed that a larger disc goes over a smaller one, and (3) the number of moves should be the least.

When the complete transfer is made, everything will be reduced to dust and the universe will end. It is not known if such a temple actually exists.

Bowls Set Transfer Game-I

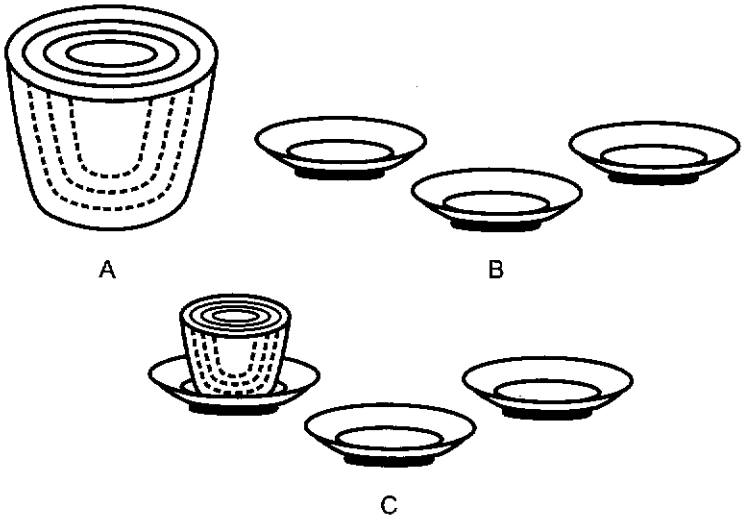


Fig. 1

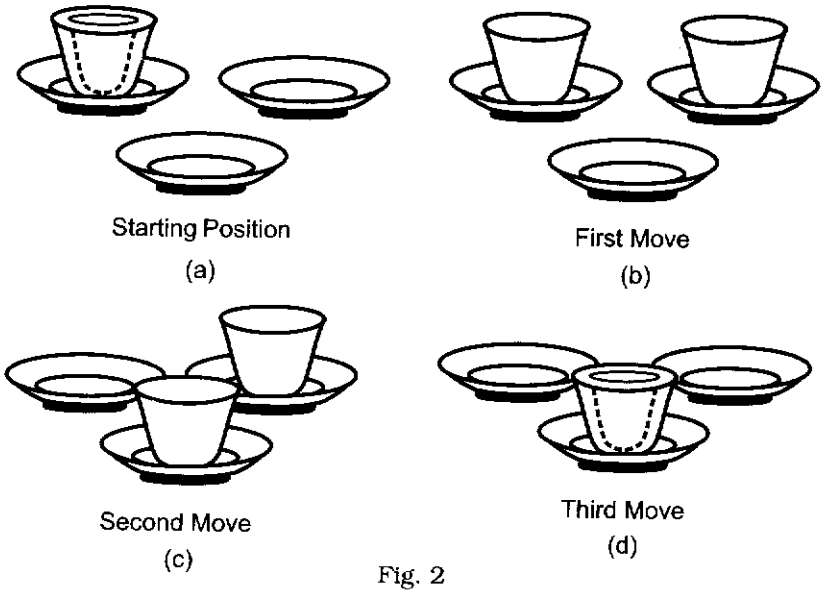


Fig. 2

Bowls Set Transfer Game-II

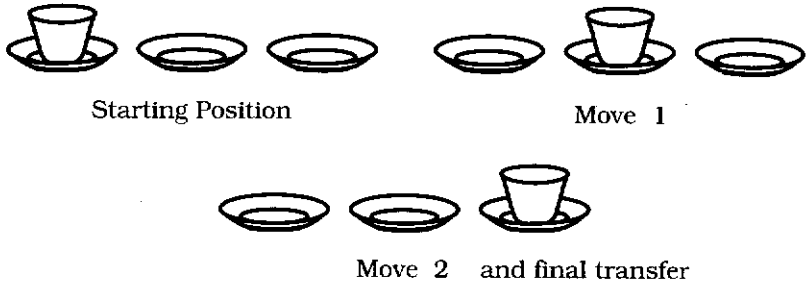


Fig. 3a

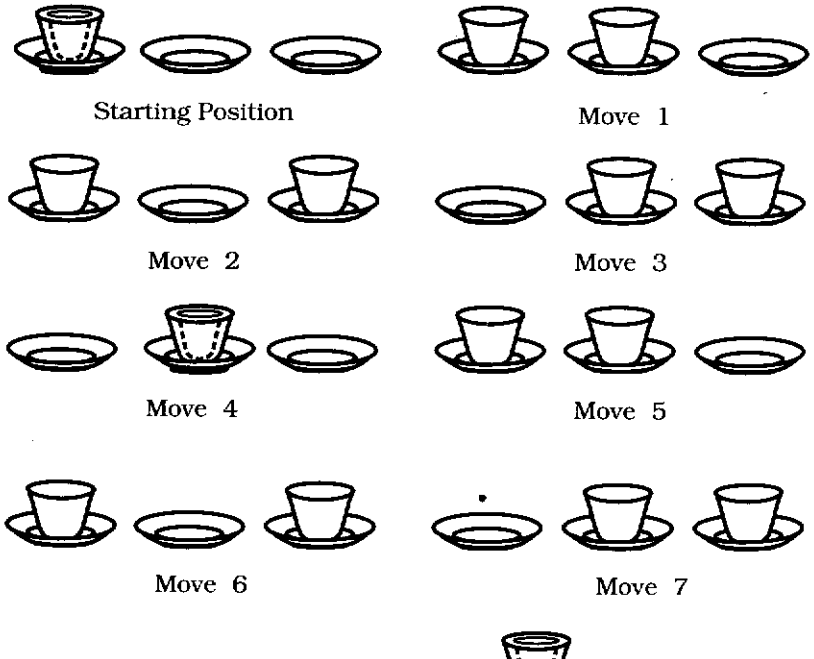


Fig. 3b

D2 – A Game with Playing Cards

1. Prologue

Many objects around us have the potential for setting up mathematical games. One such is a pack of playing cards which is so commonplace that it can be found in almost all homes in urban areas and also in many homes in rural areas.

There is an exciting maths game with playing cards which can be introduced to children in order to expose them to the art of mathematical thinking.

2. The game

A pack has four suits (sets) of thirteen cards each. In other words, there are four cards of the same value, one in each suit. All the cards except Jack, King, Queen and Ace carry number values. Let us agree to associate Jack with 10, King with 11, Queen with 12 and Ace with 13. This association is arbitrary.

Keep yourself blindfolded and give the following instructions to your child:

(1) Shuffle the cards to your satisfaction.

(2) Pick up the first card in the shuffled pack turn over and note its number (suppose it is 11). Keep it face down.

(3) Keep on picking up cards one by one, looking for the cards bearing higher numbers until 13 is got and place them face down over the first card, one by one while piling away the rest. (Cards with number 11, 12 and 13 are selected).

(4) Now pile up the selected cards (starting with the one bearing 11). This forms the first pile. The top card would have become the bottom card as each card is placed face down.

(5) Take up the pile of discarded cards and set up a second pile on similar lines (Suppose the top card now bears 8).

(6) Take up the pile of discarded cards again and set up a third pile as before (suppose the top card now bears 6).

(7) The fourth pile will consist of finally discarded cards.

(8) Of the three piles of selected cards, turn over two of them and count the cards in the fourth pile (see Fig. 1).

Now remove your blindfold, ask the child to turn over two of the three piles of selected cards, see the numbers of the top cards in the two piles and ask for the count of the cards in the fourth pile. You should be able to give the number of the top card of the remaining one of the three piles of selected cards to be turned over. That is the game.

3. The key to the solution runs thus

(1) Subtract 10 from the count of the finally discarded cards. Note the remainder.

(2) Find the sum of the numbers of the top cards of the two piles showing up.

(3) Subtract the sum from the remainder and note the result.

(4) That number is the required number of the top card of the third pile. Let the child turn the third pile over and check.

By way of illustration, consider the situation cited while spelling out the instructions. The numbers of the top cards of the three piles are 11,8 and 6 and the number of cards finally discarded would have to be 35.

Now $35-10 = 25$, $25 - (11 + 8) = 6$
$25 - (11 + 6) = 8$
$25 - (8 + 6) = 11$

4. Discovering the key

The child may ask you to explain how you are able to find out the required number. Don't just give out the key to the solution. Guide your child in discovering the key by studying the pattern emerging in a number of cases of the game. You may offer some tips.

Ask your child to tabulate the findings in a number of cases of the game as illustrated (see Fig. 2).

Numbers on the top cards of the three piles of selected cards			Number of finally discarded cards
Pile 1	Pile 2	Pile 3	Pile 4
11	8	6	35
12	9	10	41
9	9	5	33

A child will have little difficulty in discovering that *the sum of the numbers of the top cards in the three piles of selected cards is 10 less than the number of finally discarded cards.* Once the discovery is made the key to the solution easily surfaces.

5. A Project

Encourage children to get interested in generalising the game. A question that may arise will be "can't this game be played with a pack of cards having, say, 5 suits (sets) of 20 cards each ?" If so, what will be the constant to be subtracted?

The child may find that (i) a pack of cards can have any number of suits (sets), (ii) the number of piles of selected cards to be set up will be one less than the number of suits (sets), and (iii) the constant would differ with each situation. For the one suggested above with 5 suits the constant would be 16.

A teenager can be given the opportunity of *proving the existence of constant in a given situation.* For example, let the situation relate to playing cards. Algebraic approach will have to be adopted as it turns out to be the tool for investigation and explanation.

6. Explaining the game

To start with, if the top card bears the number 8, then the number of cards in the pile is 13 – 7, that is 13 – (8 – 1) or 14 – 8. To generalise, if the number of the top cards is a, then the number of cards in the pile is $14 - a$. Representing the numbers of the top cards of the other two piles of selected cards be b and c, the numbers of cards remaining in each of

the two piles are $14 - b$ and $14 - c$ respectively. The total number of cards in the three piles of selected cards is $14 - a + 14 - b + 14 - c = 42 - (a + b + c)$. So the number of cards that should be in the pile of discarded cards is $52-[42-(a + b + c)] = 10 + (a + b + c)$, showing that the sum of the numbers of top selected cards together with 10 is the number of the finally discarded cards.

7. Generalisation

A high school student can be involved in the project of final generalisation of the problem covering all such situations. If a pack of cards has k sets of r cards each, what is the constant to be subtracted? Incidentally, *the idea of parameter can be introduced to children at this juncture,* k and r have particular values in a situation whereas they vary from situation to situation. So k and r are parameters in this genre of mathematical games.

The number of finally discarded cards can be derived to be $rk\ [(r + 1)\ (k–1)\ (p^1 + p^2 + \ldots\ p^{k-1})]$ which on simplification gives $[r–k + 1 + (p^1 + p^2 + \ldots.\ p^{k-1})]$ where $p^1, p^2 \ldots p^{k-1}$ represent the numbers of the top cards in the $(k\text{-}1)$ piles of selected cards.

The attention of the children may be drawn to the clumsiness of using a, b, c in generalisation and to the elegance of using a letter with subscripts instead. Thus the parameteric form of the constant is $(r–k + 1)$ with r and k having the meanings as fixed earlier. In the case of playing cards, $r = 13$ and $k = 4$; so the constant is $(13 - 4 + 1) = 10$ which we know to be true, again in the case of the pack of cards with 5 sets of twenty cards each, $r = 20$ and $k = 5$; so the constant is $(20–5 + 1) = 16$ which also has been found to be true practically. For the purpose of the game, the numbers of top cards of $(k–2)$ piles need obviously to be given to get the number of the top cards of $(k–1)$th pile.

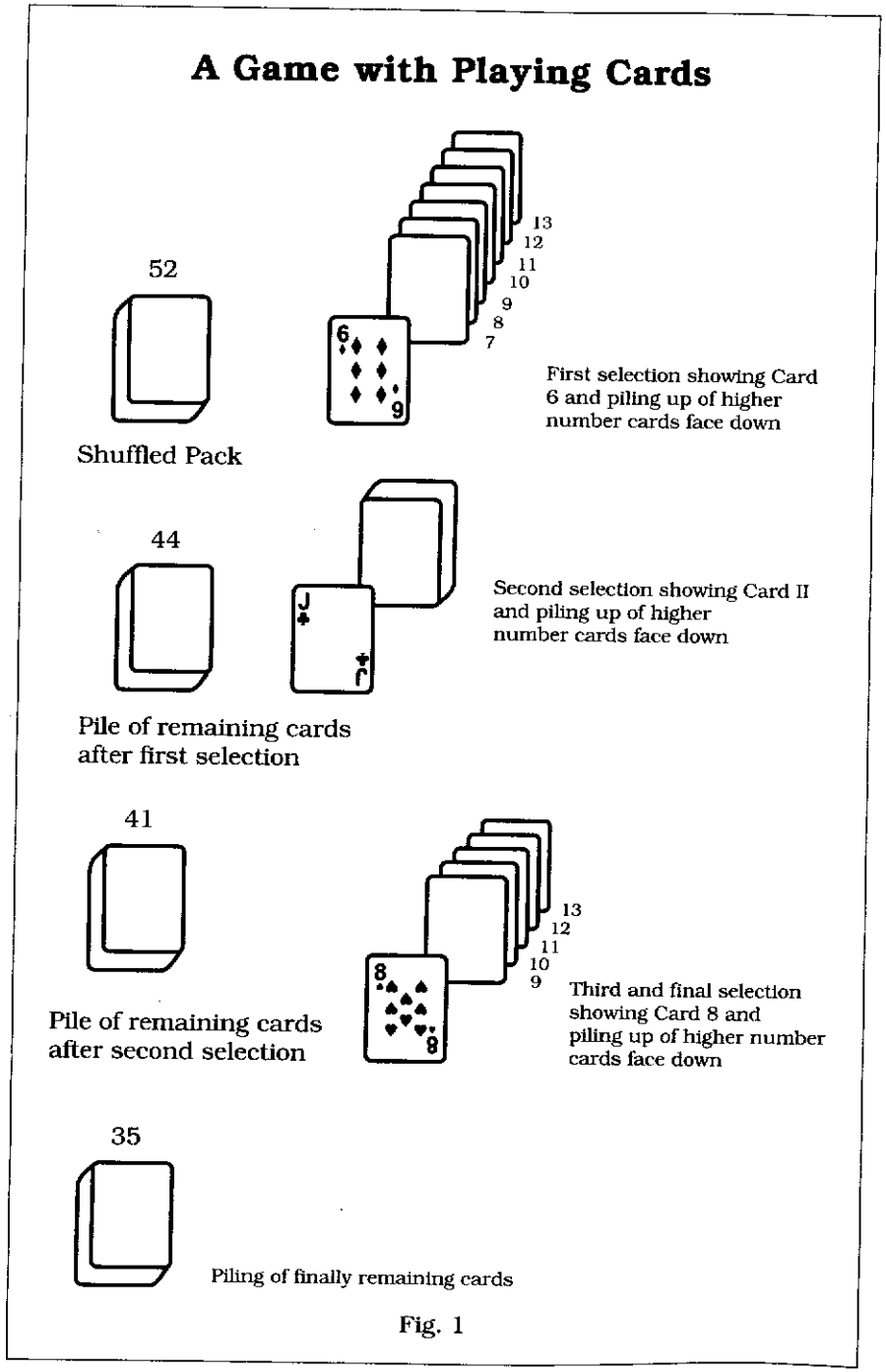


Fig. 1

D3 – Guess the Number Games

1. Prologue

Number games fascinate the young and the old. With a primary school mathematics background, scores of number games can be played and enjoyed. In a number game, usually two persons are involved. One of them initiates by asking the other to think of a number or more than one number, instructs him to do certain calculations with the number or numbers and calls for the results. Making use of the results, the initiater surprises the partner by guessing the number or numbers thought of by the latter.

Number games are generally based on simple properties of numbers, and so serve as excellent motivations to know them, thereby developing a taste for mathematics.

In super number games, there is no restriction on the number of numbers to be thought of. When they are more than two, the game is arresting, hence the adjective super.

2. The game

Let us start with the lightest of these games, lighter in the sense of the number of steps in building as well as resolving the game. Ask your child to note down three numbers less than ten in a descending order. Numbers less than ten are incidentally called digits in base ten numeration.

Tell the child not to show them to you, but to do some calculations with them according to your instructions. He should give out only the results for you to jot down if need be. The numbers can be found out by adopting the solution key in privacy.

The instruction schedule is given below along with a typical illustration. The game will be appreciated when more examples are considered:

Think of three digits in descending order	8, 5, 2
Add the first and the second	8 +5 = 13 (the first sum)

Add the second and the third	5 + 2 = 7 (the second sum)
Add the third and the first	2 + 8 = 10 (the third sum)

Tell me the three sums got and I shall tell you the three numbers thought of.

3. Key to the solution

Subtract the second sum from the first sum, i.e., in this example, 13–7 = 6. Add the difference to the third sum (6 + 10 = 16). Notice that the result obtained is twice the first number. So halve the result and give out the first number 16/2 = 8; the first sum less the first number gives the second number (13–8 = 5); the second sum less the second number gives the third number (7–5 = 2).

The three numbers thus found out can be verified.

4. Extension

A more exciting alternative to this three digit game is to ask a child to think of a three digit number and use the three digits in it to give the three sums. You can then find, as before, the three digits, and give the three digit number. The restriction of the digits to be in a descending order will of course be there. This restriction can be removed if the child is familiar with integers and knows how to handle negative numbers.

A natural extension to five digits, seven digits and so on, that is to say, odd number of digits or a number with odd number of digits can easily be made, provided one remembers that in the solution, *the sums are to be subtracted and added alternately, to get twice the first number.*

An interesting question would naturally be raised at this juncture. Cannot the number game be played with four digits, six digits and so on, that is to say, even number of digits or a number with even number of digits? Fortunately the answer is in the affirmative. But there is a slight variation in the instructions to be given and in the method to find the solution. Study the illustration given below

Instruction	*An illustration*
Think of four digits in descending order	6, 4, 2, 1
Add the first and the second	6+4=10 (the first sum)
Add the second and the third	4+2 = 6 (the second sum)

Add the third and the fourth	2+1=3 (the third sum)
Add the fourth and the second	1+4=5 (the fourth sum)

(Note: the first is not to be added here)

Set aside the first sum (10); subtract the third sum from the second sum (6–3=3); add the difference to the fourth sum (3 + 5 = 8). *Notice that the result obtained is twice the second digit.* So halve the result and remember it (8/2 = 4).

The first sum less second digit gives the first number (10–4 = 6). The second number has already been obtained (4). The second sum less the second digit gives the third digit (6–4=2); The third sum less the third digit gives the fourth digit (3–2=1).

The four digits can be verified.

The game can be played with a four digit number, instead of four digits. Also instead of uni-digit numbers, multi-digit numbers, can be used.

5. Explanation

The game would not be mathematically worthwhile if the property on which it is based is not understood. Assuming the three numbers to be a, b, c, their sums, two by two, will be $(a + b)$, $(b + c)$, and $(c + a)$. Now it can be seen easily that $(a + b) - (b + c) + (c + a) = 2a$, and this can be extended to any odd number of numbers. The game can also be looked upon as a recreational approach to simultaneous equations.

A Game of Number Prediction-I

Player

4	2	7	8	
6	9	6		(4 + 2; 2 + 7; 7 + 8 = 15; 1 + 5)
6	6			(6 + 9 = 15; 1 + 5 = 6; 9 + 6 = 15; 1 + 5 = 6)
3				(6 + 6 = 12; 1 + 2)

Fig. 1

$a \qquad b \qquad c \qquad d$

$a+b \qquad b+c \qquad c+d$

$a+2b+c \qquad b+2c+d$

$a+3b+3c+d$

Fig. 2

D4 – A Game of Number Prediction

1. Prologue

Each number has its personality and the joy of mathematics lies in knowing it to the extent that more mention of a certain number evokes in one's mind many of its beautiful characteristics. It is said that to Ramanujam every number up to 10,000 was his personal friend.

Arithmetic will cease to be dull and tedious, if children get exposed to the personality of each number, at least up to a hundred. Without knowing the personality of single digit numbers, primary school mathematics will be incomplete, even though children may have been helped successfully to go through the grind of arithmetical operations and solving of word problems.

2. The personality of 9

Nine is the most fascinating single digit number. Anyone running through its multiples 09, 18, 27, 36, 45, 54, 63, 72, 81, etc. cannot but be alive to the patterns, some of which are on the surface and some deep. Digits keep decreasing in the units places, whereas digits keep increasing in the tens places, but the sum of the digits in each multiple turns out to be 9. This gives a clue for finding whether any number can be divided by 9 without a remainder.

For example, take the number 783, What is the digit sum? It is 7 + 8 + 3 = 18. 783 is divisible by 9 if 18 is divisible by 9. To see if 18 is divisible, find the digit sum again, It is 1 + 8 = 9 and since nine is got ultimately, 783 is divisible by 9. Let us check and see: 783 = 9= 87 R 0 (quotient 87 and remainder 0). Many such examples suggest the rule that if the ultimate digit sum or digital root of a number is nine, the number is divisible by 9. What does it mean, if the ultimate digit sum is not 9 ? Consider the number 23, for instance, 23 = 9 = 2 R 5. But the remainder 5 can be got simply by finding the digit sum: 2 + 3 = 5. Many such examples suggest the rule *that if*

the ultimate digit sum of a number is other than 9, then the ultimate digit sum gives the remainder when the number is divided by 9.

There are some beautiful shortcuts in finding the digit sum. Consider a two digit number, having a 9 for one of its digits. Let us take the example 93. Its ultimate digit sum is 3 (9 + 3 = 12, 1 + 2 = 3). So while finding out the ultimate digit sum of a number, all 9s can be ignored. Not only that, whenever 9 turns up as a partial sum while summing the digits, it can be ignored and summing started afresh from that stage onwards. Consider, for instance, 612593. Its digit sum is simply 5 + 3 = 8. (How? 6 + 1 + 2 = 9; ignore 9; ignore 9; 5 + 9 = 5, 5 + 3 = 8). Also when a two digit sum appears as a partial sum, summing of digits of the partial sum can be done to get a single digit partial sum. An example will help. The ultimate digit sum of 8768 is 2. (How? 8 + 7 = 15, 1 + 5 = 6; 6 + 6 = 12, 1 + 2 = 3; 3 + 8 = 11, 1 + 1 = 2).

The behaviour of 9 can be exploited in setting up an interesting number game. It is better to familiarise children with the game first, before they begin exploring its ramifications and their rationale. The game can be played with a multi-digit number of any number of digits. Let us start with a four digit number to avoid bewilderment and then prepare for handling multi-digit numbers with a greater number of digits.

3. The Game

The game needs at least two persons. Let one of them be a child and the other yourself. Ask the child to write a four digit to start with, take its digits in pairs from the left end, keeping the order of digits unchanged, find the ultimate digit sum of each pair and write down the resulting digits one by one. This will yield a three digit number. Ask the child to continue the same process of summing digits in pairs to fix the next two digit number and finally the one digit number.

The prediction game is to give out in advance the final one digit number by inspection of the four digit number. An example is given (see Fig. 1).

The short cut in predicting the final digit shows itself: *Add the end digits, take thrice the sum of the middle digits, add the results and find the ultimate digit sum.* Taking 4278, the short cut will be (4 + 8) + 3× (2 + 7) = 3 + 9 = 3 (= means equality in a certain sense).

Change roles with the child and play this game with many other four digit numbers.

The curiosity in children will compel them to raise a number of interesting questions such as :

(1) How to find the sequence of numbers used for multiplying the successive digits of a given multi-digit number ? (call them digit multipliers)

(2) Is it possible to predict the number got at any stage of working done by the player?

To find the answers to these questions,

Children should be helped to spend some time in generalised arithmetic or school algebra. Let us help children to know how the digit multipliers are found for a four digit number. We shall, in general, indicate a four digit as abcd (Note that $a\ b\ c\ d$ does not mean here $a \times b \times c \times d$ but $1000a + 100b + 10c + d$ as is a four digit number 2 3 7 4). In terms of the players role, let us work out the derivations (see Fig. 2).

4. Digit Multipliers

The starting number has 4 digits. So use the digit multipliers 1, 3, 3, 1 respectively for 4, 2, 7, 8 in order, multiply and find the ultimate digit sum. 4 × 1 + 2 × 3 + 7 × 3 + 8 × 1 = 4 + 6 + 21 + 8 = 1 + 3 + 8 = 4 + 8 = 3.

$a\ \ b\ \ c\ \ d$
$a+b\ \ \ b+c\ \ \ c+d$
$a+2b+c\ \ \ b+2c+d$
$a+3b+3c+d$

This shows how algebra can be looked upon as X-ray or scanner of arithmetic. The last derivation giving the ultimate digit is $a+3b + 3c + d$ or $1a + 3b + 3c + 1d$. This reveals that the digit multipliers respectively for the four digits a, b, c, d are 1, 3, 3 and 1. Since $a + 3b + 3c + 8d = (a + d) + 3 (b + c)$, the short cut is also easily explained.

Now children can set up algebraic forms for multi-digit numbers, with numbers of digits ranging from say 2 to 10, and discover the digit multipliers for each case. Check whether children get the following. Note that ultimate digit sum is to be used whenever two digit multipliers appear.

5. Elegance

If children are familiar with integers, they can replace digits, 5, 6, 7, 8, 9 respectively by –4, –3, –2, –1 and effect a very elegant simplification.

A Game of Number Prediction-II

Number of digits	List derivation representing the final digit	The sequence of digit multipliers
2		11
3	$a-b$	121
4	$a-2b-c$	1331
5	$a-3b-3c-d$	14641
6	$a-4b-6c-4d-e$	151151
7	$a-5b-c-d-5e-f$	1662661
8	$a-6b-6c-2d-6e-6f-g$	17388371
9	$a-7b-3c-8d-8e-3f-7g-h$	181272181
10	$a-8b-c-2d-7e-2f-g-8h-k$	1993993991
	$a-9b-9c-3d-9e-9f-3g-9h-9k-l$	

Fig. 3

D5 – Discover Fun with Bus Tickets

1. Prologue

Numbers appear in different contexts and in different arrangements. Any arrangement when observed closely and carefully makes even a primary school child recall rich associations and properties based upon his background acquired in the classroom. When encouraged to do a little exploration, he gets stimulated to see more and more. To uncover the hidden relations, links and associations in numbers is a source of joy. When he is helped to distinguish a shallow pattern from a deep one, his joy is all the more.

Some bus tickets carry stage numbers on the left and right margins for the up and down journey on a route. The numbers appear in ascending and descending orders and so provide an opportunity for children to examine link patterns. A typical bus ticket is given below (see Fig. 1).

Of course, the number arrangements are not the same in all bus tickets as the number of stages cannot be the same for all the routes and the conventions in indicating stages vary. The number arrangements have to be observed first to identify the kind.

2. Patterns detection

(1) Some questions naturally arise in children's minds. For instance is it possible to give the corresponding number on the right column when a number on the left column is mentioned and vice versa? This involves discovering the pairing pattern. A hint to compare number-pairs by their sum, difference, product or quotient may be given. Children soon see that a number on the left (or right) is paired with its corresponding number on the right (or left) in such a way that their sum is always the same or constant; here it is 16. If the number in the left margin is called L and that on right R, the child can be observed stating the relation; L+R=16. Incidentally, the

arrangement here give the fifteen complementary addition facts of the number 16:

(16 = 1 + 15 = 2+ 14 =... 15 + 1)

(2) Finding sum without adding: The sum of all the numbers in both the margins is easily given as 15 × 16, as there are 15 pairs, each pair totaling 16 and so the sum of all the numbers in either margin is half the product, that is, 15 × 16/2. Considering such arrangements with more numbers, the method of finding the sum of a given set of all counting numbers starting from 1 and stopping at some stage surfaces and it turns out to be half the product of last number counted and its next higher number. The child when exposed to this for the first time gets overwhelmed by the power of mathematical thinking.

(3) A child who knows how to identify odd and even numbers sees, in the matching arrangement seen on the bus ticket chosen, odd matched with odd and even with even. This is not true of every arrangement. By examining more bus tickets or by making up such number pair arrangements on one's own, the child discovers that odd-odd and even-even matching occurs when the last number in the left margin or the first number in the right margin is odd. If that number is even, the pattern changes to odd-even and even-odd matching.

If the last number in the left margin is given, is it possible to guess the number of odd-odd matchings in the arrangement? By studying more arrangements of this type, the child discovers that the number of odd-odd matchings is the same as the middle numbers. In the bus ticket considered, the number of odd-odd matchings is 8 which is seen to be the middle number in the sequence.

(4) How to find the middle numbers? In case the last number is odd, then at one stage the left number is the same as the right number and that number is the middle number. It is also the mean or the average of all the numbers on either margin.

If numbers on either margin end with an even number, there is no middle number, but an interesting pattern occurs. Consider numbers 1 to 16 on the left margin and 16 to 1 on the right margin. Two pairs of side by side (consecutive) numbers are observed to match in reverse order: 8, 9 in one respectively with 9, 8 in the other. These become two middle numbers.

(5) The two margins are parallel. Ask children to join pairs of associated points, left point 1 with right point 1, left point 2 with right point 2 and so on. A pleasant surprise awaits the children, as they will soon see that all the lines drawn pass through one point. To put it in mathematicians language, the lines are *concurrent*, in other words they have a common junction. Each line segment is cut into two equal parts at the common crossing point and the shortest line segment is the one joining mid-points on both the margins (see fig. 2).

(6) It is interesting to count the number of triangles formed and the relation the number bears to the number of points on either margin. A middle school student should be able to identify pairs of triangles having the same shape and size (in other words, congruent triangles).

(7) Writing the differences obtained by subtracting each number on the left margin from its corresponding number on the right margin, children see a sequence of positive and negative numbers emerging.

(8) When the products are struck, the products are seen to rise, reach a maximum and then fall in value.

3. Conclusion

Simplicity and profundity are intertwined in mathematical investigations and it would do children good if they are helped to get the message as early as possible.

Discover Fun with Bus Tickets

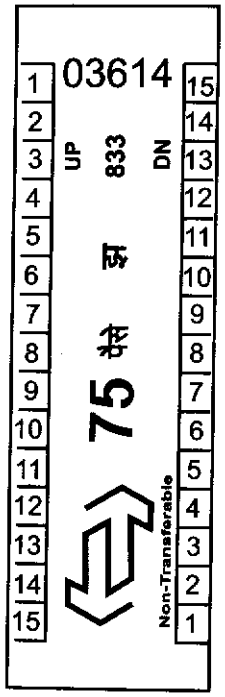


Fig. 1

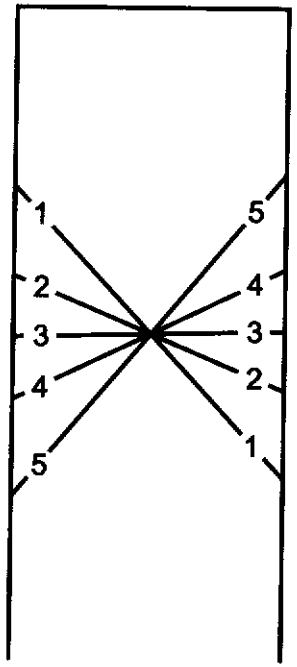


Fig. 2

D6 – Mathematics with Match Boxes

1. Prologue

Children in both rural and urban areas are quite familiar with match boxes. Some are avid collectors of match box labels. Children of upper primary level can have an interesting and valuable experience, if their attention is directed to mathematical relations that match boxes can show up.

To start with, let children find out the number of ways in which a given number of match boxes of the same brand can be arranged to form cuboidal (rectangular solid) shapes and take the first step in realising that different shapes can have the same volume. Any box in the same brand can serve as a unit of volume, when volumes of solids made with the same brand of boxes are to be compared. Though these different shapes in each case have the same volume, their surface areas are different.

2. Study of volume and surface

Incidentally a comparative study of surface areas is related to saving of packaging material. It is worth investigating to find out if there is a functional relation between the number of cuboidal arrangements and the number of match boxes chosen. If so, the question of finding the formula would be quite challenging.

Match boxes are all cuboidal, though they are available in slightly different sizes. Ask children to collect four match boxes of each brand and test the relations between the length, the width and the thickness of a box in each brand. This can be done without using any graduated ruler.

A match box has three pairs of identical faces, large, medium and small in area. We shall call the largest face Top, the medium face Side and the small face End. Two Sides make one Top and 3 Ends make one Top (see fig. 1). Help children to see that this holds good approximately in almost all brands. A match box having these area relations exactly is a standard

one (it is desirable that ISI stipulates this). A standard match box has length, width and thickness and with thickness as the unit of length, the dimensions of a standard match box can be given as three units, two units and one unit. By actual comparison of areas, children find no difficulty in stating that End is 1/3 of Top. So the surface area of a standard match box can be given as 3 2/3 Tops. Top serves as the unit for surface area. It is important to note that volume and surface area are not found here by the use of conventional formulae. Formulae can be used if so desired to give the measures in conventional units.

3. Packaging ways

Children love to enumerate by experimentation the various ways in which a given number of match boxes (of the same size or brand) can be arranged for cuboidal packaging.

(1) Taking *two* match boxes of the same brand they can be arranged in three ways: one over the other and one by the side of the other in two ways. The dimensions are 1, b, $2t$; 21, b, t; 1, $2b$, t.

(2) Taking *three* match boxes of the same brand, in how many ways can they be arranged? Children discover that the number cannot be less than three in any case, dimensions here being given as 1, b, $3t$; 31, b, t; I, $3b$, t. There are no more ways.

(3) Taking *four* match boxes of the same brand, the familiar three ways giving the dimensions as 1, b, $4t$; 41, b, t; 1, $4b$, t are first taken note of. By practical experiment, three more ways are seen to surface. Their dimensions are 21, $2b$, t; 21, b, $2t$; 1, $2b$, 21. Since the relations that End and Side bear to the Top are known, one more way, the sev‹ ·th shows up, the dimensions being, $b + 2t$, $2t$ or b, See figure 2.

4. Saving packaging paper

Ask children to compare the surface areas of the various combinations and specify the combination that has the least surface area so that packaging paper can be saved.

5. Project

An exciting project can be seen to catch the fancy of children. Is it possible to give the number of combinations for a given number of match boxes? This is an excellent instance at elementary level to introduce mathematical modelling in which forming permutations of the triple factors of the number of match boxes and using the triple factors as multiples of the dimensions will be found relevant and fruitful.

Mathematics of Match Boxes

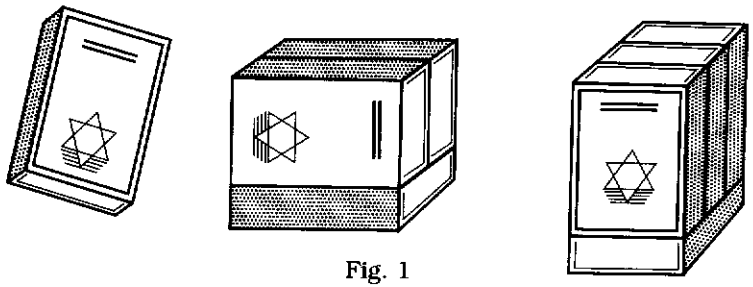


Fig. 1

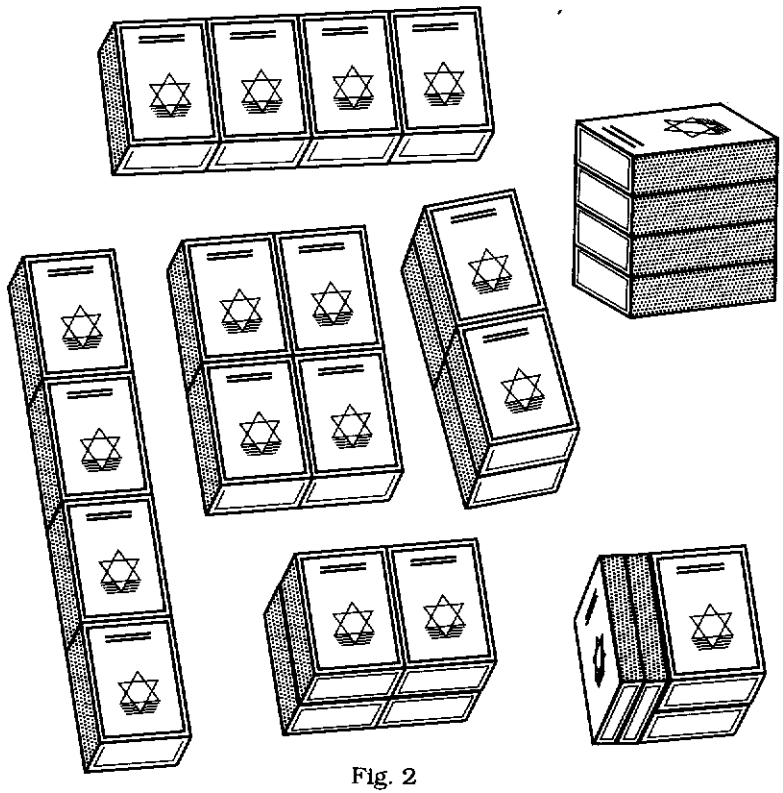


Fig. 2

D7 – An Inverting Game with Coins

1. Prologue

Games which show up shifting patterns of moves have great attraction for children. Such games provide high motivation in developing a mathematical attitude, as children get an opportunity to find their guesses going wrong and feel compelled to change their viewpoints. This brings about the realisation that generalisation is a serious thing in life and more so in mathematics.

2. The game

One such game centres around setting up a triangular array of coins. In a triangular array of objects, each row has one more object than the preceding row and the first row starts with just one object. Also the number of objects in the row alone is the same as the number of rows (see figures).

Ten paise coins or Carrom board coins are quite handy for playing this game. Since different guesses would be made, the game is better played by a group of two or three children.

(1) Ask children to set up a *two row triangle with three coins.* It is easily seen that by shifting just one coin, the triangle gets inverted (see Fig. 1).

(2) Taking next *the three row triangle with six coins,* it requires moving two coins only, to invert the triangle (see Fig. 2).

Having found so far that the number of coins to be moved is one less than the number of rows, children are seen to hazard the guess that for inverting the *four row triangle with ten coins,* moving of just three coins would be enough and it turns out to be true (see Fig. 3). But soon it is realised that this patterns fails in the case of the next set up of the five row triangle. The least number of coins to be moved is not four, as should plausibly have been according to the pattern so far guessed, but five (see Fig. 4).

The itch to make a true guess this time for the *six row triangle drives a child* to review the sequence of the number of moves of coins against the number of rows and plump for a number.

No. of rows	2	3	4	5	6
No. of moves	1	2	3	5	?

Viewing the pattern to be 1 + 2 = 3 and 2 + 3 = 5, some children can be seen to guess the required number of 1 moves to be 3 + 5 = 8, whereas some other children opt for 6. Let children set up the six row triangle and actually try to invert it by the least number of moves. They discover with a bang that both the guesses are not correct, as the least number of moves turns out to be 7.

Note:

Coins that remain unchanged in position are indicated by circles without dots and coins that are to be moved and that are moved by circles with dots.

3. Strategy

Allow children to play this inverting game going up to triangles with 12 rows. Watch a child suggesting a speedier way of determining the number of coins to be moved by making out a triangular array of dots, using a tracing paper to copy the array and inverting the trace over the array in such a way that most of the dots overlap. By moving the trace a little up or down, the required least number of dots to be moved is easily determined by counting those which are not overlapped. This can be called inverted trace placement or ITP technique. This kind of creativity to strike a short cut is often seen to surface when the pressure to avoid tedium mounts.

The next step would naturally be to tabulate the observations as follows:

No. of rows in the triangle:	1	2	3	4	5	6	7	8	9	10	...
The least no. of coins to be moved to invert the triangle:	0	1	2	3	5	7	9	1	15	18	...

4. Search for the hidden pattern

Children could be seen engaged in animated discussion, tossing one guess after another. Someone succeeds in spelling out the pattern to the excitement and relief of the group.

0	1	2	3	5	7	9	12	15	18	...
	1	1	1	2	2	2	3	3	3	...

Leave the first, keep adding 1 and get 1, 2 and 3. Start from 3, keep adding 2 and get, 5, 7 and 9. Start from 9, keep adding 3 and get 12, 15 and 18 and so on. The pattern gallops at every third step. *What a fascinating pattern.*

Children at this stage, feel intuitively secure to find and say that to get the least number of coins to be moved for an 11 row triangle, four is to be added to 18 to get 22 and for 12 row triangle it will be 22 + 4 = 26. This can be verified by the ITP technique.

5. Formula

Finally it would be natural to seek a formula as the table that has been drawn up does not seem to clear the way. Suggest to children to draw up an associate table, giving the number of coins in each triangle and the least number of coins to be moved.

No. of coins:	1	3	6	10	15	21	28	36	...
The least number of coins to be moved:	0	1	2	3	5	7	9	12	...

This table exposes the relation between the number of coins and the least number of coins to be moved for inversion of the triangle. In numerous cases, the least number of coins to be moved is seen to be 1/3 the number of coins in the triangular array. But there are cases when division by 3 leaves remainders and the remainders in such cases have to be dropped. This covers the cases of exact division also as the remainders to be dropped are simply zeros. It would be appropriate at this stage to introduce the mathematicians symbol of square brackets to indicate dropping of the remainder and retaining of the quotient alone.

Examples $\frac{10}{3} = 3;\ \frac{23}{3} = 7;\ \frac{3}{3} = 1;\ \frac{12}{3} = 4$

To get the general formula, children may denote the number of coins by N and the least number of coins to be moved to invert the triangle by M. then

$$M = \frac{N}{3}$$

If children know that the relation between the number of coins N and the number of rows R (say) is given by

$$N = \frac{R(R+1)}{2}$$

then the formula for getting the least number of moves can be got straight from the number of rows itself as given below

$$M = \frac{R(R+1)}{6}$$

6. Project

Giving proof for this is most challenging and the gifted can get an opportunity to exhibit their mathematical talents.

An Inverting Game with Coins

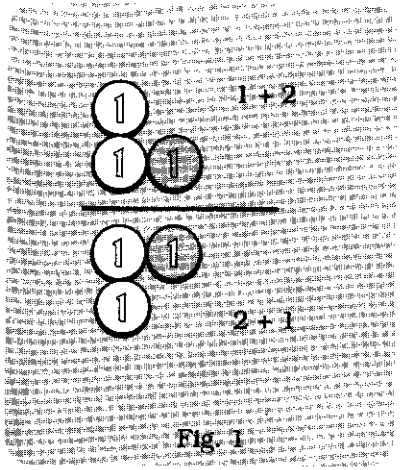


Fig. 1

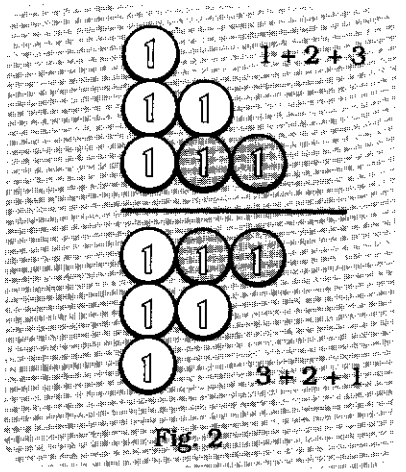


Fig. 2

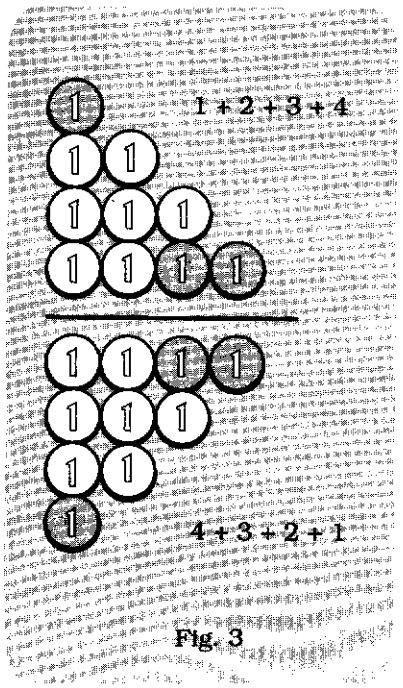


Fig. 3

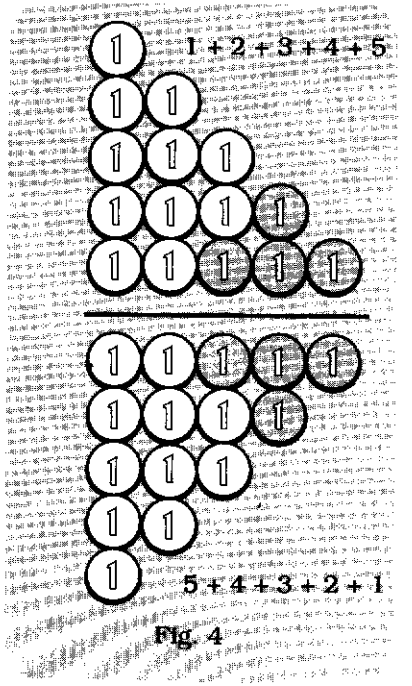


Fig. 4

D8 – Number Games

1. Prologue

Children love novelty as it provides excitement. This is missing in our mathematics lessons where they have to do routine things introduced in a prosaic manner. If only teachers take interest in suggesting projects involving non-routine explorations arising out of regular themes covered in the class, the school will contribute a lot to the mathematical growth of its children. This requires getting rid of the syndrome of expecting children to do only things that they are taught. The disinclination to encourage children to tackle problems on their own should go.

Are not children individualistic? Do they not love to do non-routine, innovative and unusual things to proclaim their individuality? Recognition of this need should be acknowledged. An instance is suggested to initiate this line of thinking. The non-routine project that is presented below could be assigned to children of primary schools once they have acquired the ability to build numbers with given digits and do addition of whole numbers.

2. Addition without addition

(1) By taking a digit and repeating it, numbers represented by multi-digit numerals can be identified. Ask children to start with say 2, and build numbers by repeating it once, twice, thrice, and so on. They would get 2, 22, 222, 2222, etc. Ask them to add them progressively, giving the sums at each stage. The sums will be respectively 2, 24, 246, 2468 and so on. Challenge them to discover the pattern of digits in the sums by comparing them with digits in their corresponding addends. This discovery would endow children with the strategy to strike the sum of addends got by progressive repetition of a digit to any specified stage without writing the addends at all. Children have to be, of course, careful when the stage is reached which involves carrying figures while striking the sum.

(2) There is another project more exciting than the one suggested above. Given more than one digit, children may be asked to form numbers with the given digits in two stages, (i) without repetition of digits, and (ii) with repetition of digits and find their sums separately and then together. Ask children to discover the relation that the digits constituting the sum bear to the digits constituting the addends. Children should be able to strike the sum without writing out first all the numbers of addends that can be formed and then adding them patiently. This experience endows them with a sense of power which they would like to exhibit to participating audiences in club meets and even on school anniversaries for that matter.

If, for instance, children start with two digits, say 2 and 4 the numbers that can be formed without repetition of digits are 24 and 42 and with repetition of digits 22 and 44. Their sums are 66 when taken separately and 132 when taken together. Children compare the sum of the digits chosen and the digits in the sums obtained and discover the strategy to strike the sum without writing the addends and adding them. Since the number of digits chosen here is two, there can be single digit numbers forming the third stage. The sum of all the numbers in three stages is 138 which also can be struck without writing out the numbers.

(3) Taking three digits next, children will be able to form numbers in four stages (i) without repetition of digits, (ii) with repetition of digits, (iii) two digit numbers without and with repetition of digits, and (iv) single digit numbers. Let children count the numbers in each stage. Tell them that they should have 6 numbers in stage (i), 21 numbers in stage (ii), 6 without repetition and 3 with repetition in stage (iii), and 3 single, digit numbers in stage (iv). Children would be wondering how you could give the count without spelling out the numbers, at different stages. This would leave them at the threshold of another

project leading to an exciting branch of mathematics, combinatorics. The sum of numbers at any stage can simply be struck without writing them at all, once the pattern of relationship between the sum of the chosen digits and the digits in the sum struck is discovered.

3. Project

A high school student can use algebra and develop the formula for striking the sum with any number of chosen digits. Inclusion of zero among digits will make the project more exciting.

E. Investigation (Non-routine Problems)

E 1 – Time for Investigation

1. Prologue

There is no topic in school mathematics which does not lend itself to numerous investigations by children. No syllabus or teaching can cover even all the elementary aspects of a topic set down for study at school. What is needed is receptivity on the part of children to questions that arise in their minds and encouragement on the part of adults (parents and teachers) for creating an air of expectancy and appreciation for children who seek answers.

2. Rectangles into a square

Who does not know a rectangle and a square? Who does not know how to build a rectangle with unit squares? Therefore there arises the question of building a square with rectangles of the same size. The 'investigation' centres round finding the minimum number of equal-sized rectangles to be chosen to build a square and predicting the side of the square that could thus be built.

(1) To start with, rectangles with length and breadth in *natural number* units are chosen. Use of square ruled sheets for making multiples of rectangles of a given size will be found helpful. Let the sizes be 3×2, 4×3, 9×3, 8×2, 8×6 and 12×8. Unless a child is endowed with spatial *intuition* of a high order, the child cannot but indulge in some experimentation to build squares with rectangles of the given size and find the minimum number of rectangles required.

A few typical illustrations from the work done by children in this investigation are presented, followed by tabulation of results.

1	b	a	m minimum number of rectangles
3	2	6	6
4	3	12	12
4	2	4	2
9	3	9	3
6	4	12	6
8	6	24	12

The table shows that it is easy to predict the minimum number of rectangles and the side of the built square in cases where the measures of the length and the breadth are consecutive numbers (more generally, relatively prime numbers) or pairs of numbers with the greater one being a multiple of the other. When the two measures are multiples of 2, 3 etc., the prediction becomes rather difficult. By comparing column 1 and column b numbers with column a numbers (a representing the side of the built square), children are excited to find that the side of the square built is simply the LCM (the least common multiple) of the 1 and b numbers (natural ones).

3. The formula

The next question is to predict the minimum number of rectangles required. By examining the pictures or the table, the children discover that the minimum number of rectangles is simply the square of the LCM of 1 and b (LCM of 1 and b gives side a of the square) divided by the product of 1 and b. When written in symbols, it becomes the formula for finding the minimum number (m) of rectangles:

$$m = \frac{a}{1} \times \frac{a}{b} \text{ or } \frac{a^2}{1b} \text{ where } a \text{ is the LCM of 1 and } b$$

Here is an instance about a non-routine application of LCM.

4. Epilogue

A healthy tradition to nurture investigative attitude in children needs to be built up by setting apart investigational time during weekends.

Time for Investigation

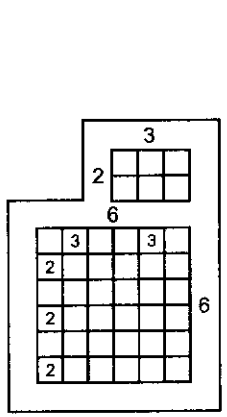


Fig. 1

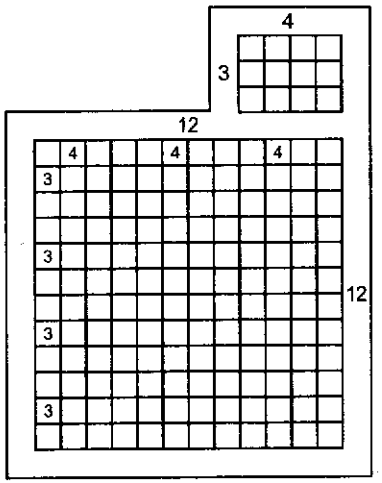


Fig. 2

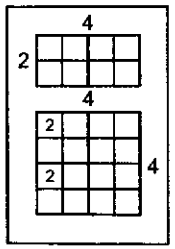


Fig. 3

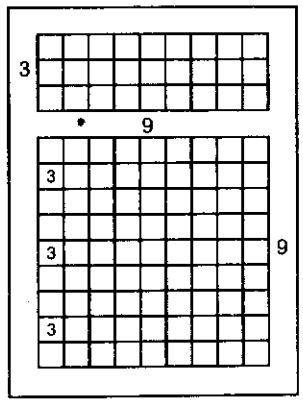


Fig. 4

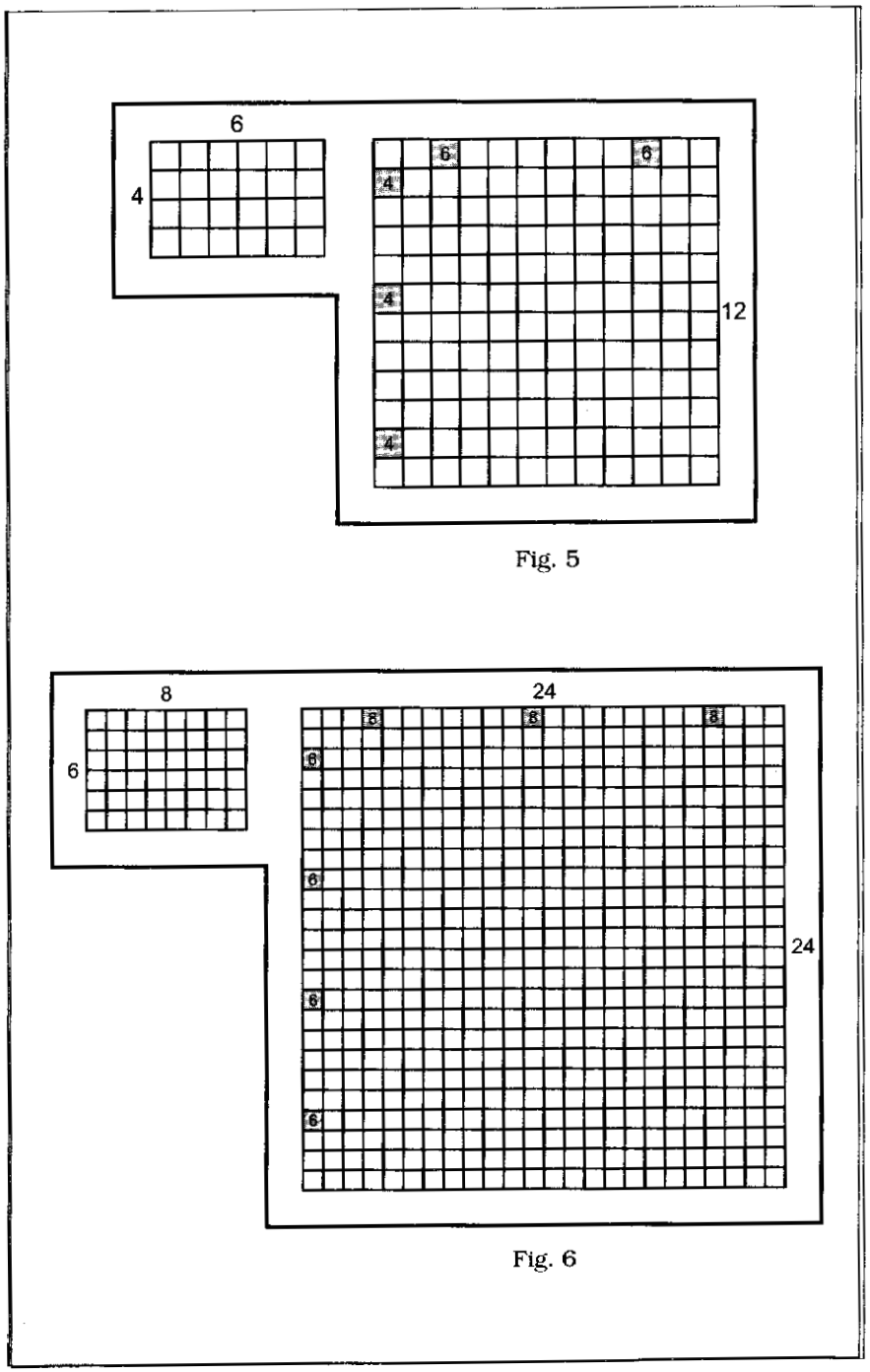


Fig. 5

Fig. 6

E 2 – Prods to Mathematical Thinking

1. Prologue

The most important objective in mathematics education is to promote thinking. If an open-ended question can be suitably chosen so as to fall within the competence of a child for response, the objective can be promoted and achieved easily. Unlike other subjects in the curriculum which stress more on facts than on concepts, mathematics offers a rich source of open-ended questions, right from the nursery years. Open-ended questions in mathematics should be sharp ones which do not diffuse. Therefore what an educator, a parent or a teacher has to do is to be aware of such a situation and make sure that every time a child succeeds he learns a concept. The role of open-ended questions in mathematics education has not received the attention it deserves in our schools. Consider the stage when a child learns the elementary concept of the perimeter of a geometric figure starting with the perimeters of a rectangle and a square and is taught how to spell out the perimeters in the forms of formulae (see Figs. 1 and 2).

2. Open-ended Questions

Assuming that children are exposed to the experience of building, identifying and naming polygonal shapes, triangles and quadrilaterals in particular, with sticks, suitably chosen, pose the open question of identifying shapes for a given perimeter expression such as $a + b + c$, $a + 2b$ or $3a$. *The inverse of a direct question often turns out to be an open-ended question, showing an easy way to frame it.* The child's conceptual development can be effectively watched and tested when the child suggests types of following figures. If the child can tackle the questions without the actual use of sticks, the child can be assessed to exhibit signs of exceptional ability. Tutoring should be avoided at all costs, since no achievement test for which children are coached will normally includes such questions.

Given below are the responses of children, for this question (see figures 3, 4 and 5).

3. Extension

The question becomes really exciting and deeper when the expression giving the perimeter has more than 3 letter symbols. Ask children to identify shapes having the perimeter given by $a+b+c+d$, $a+2b+c$, $2a+2b$, $a+3b$, and $4a$. Naming is secondary and so need not be insisted upon initially, as it involves more of recall than of thinking. It is interesting to watch how many children do not stop with convex (no caving in at any corner) figures alone but also cover non-convex (caving in at some corners) figures.

4. From perimeter to area

If expressions for areas are taken up for forming open-ended questions, children really get an opportunity to exhibit their range of understanding and acquisition of conceptual richness. In other words, they would indeed get out of such an encounter and there will be an upswing in their appreciation of mathematics.

Prods to Maths Thinking

$P = 2 (1 + b)$

Fig. 1

$P = 4a$

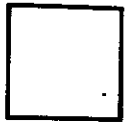


Fig. 2

$P = a + b + c$

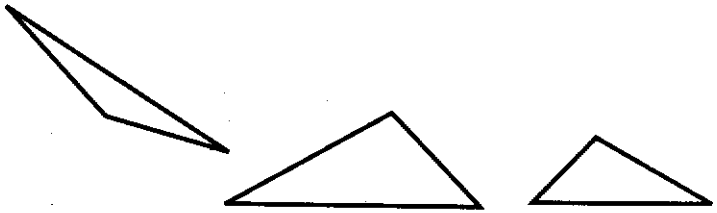


Fig. 3

$P = a + 2b$

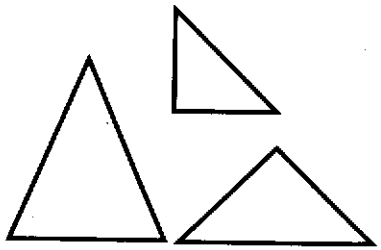


Fig. 4

$P = 3a$

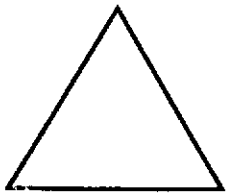


Fig. 5

E 3 – Investigational Spree

1. Prologue

The Cockcroft Committee of Inquiry into the Teaching of Mathematics in Schools in England and Wales has, in its report published in 1982, almost spelled out the criteria, six in number that should govern successful teaching at all levels. They are (1) exposition by the teacher, (2) discussion between teacher and pupils and between pupils themselves, (3) appropriate practical work, (4) consolidation and practice of fundamental skills and routines, (5) problem solving, and (6) investigational work. It would be worth asking how many schools would be ready to plan their schedule of work to meet these criteria, leaving alone the question of measuring up to the level of these of expectations.

It is often and widely felt that children are too young or inexperienced to do any investigation on their own. But this has been found to be incorrect by those who have encouraged children to pursue their questions of enquiry and discover the answers. In an age when computers and robots are taking over routing and tedious tasks, computational as well as logical, it is considered a criminal waste of time to train children to acquire merely skills at the expense of nurturing and developing their potential for creativity and originality. The easy way to promote this desirable feature of educational endeavour is to include, in the scheme of things, children's involvement in investigational programmes and opportunities for exposition to the public. What should be the quality or standard of the investigations pursued by children? In the words of Cockcroft Committee report, 'investigations need be neither lengthy nor difficult. At the most fundamental level and perhaps on most of the occasions they should start in response to pupils' questions, perhaps during explanation by the teacher or as a result of piece of work which is in progress or has just been completed'. Without investigational work, children will have to wait for long to get a feel of

mathematics and develop a taste for mathematical thinking on their own.

An instance is presented here to show how children become curious about certain things happening in a familiar situation and how, when encouraged, they exercise their mathematical 'muscles' and make remarkable discoveries of a non-trivial character.

2. A non-routine problem for study

The most familiar geometric figure right from the primary years is the rectangle. When a rectangle built up of equal sized squares is taken, that is to say, when the length and breadth of a rectangle are given in natural number units and one of its diagonals (lines joining opposite corners) is drawn, the diagonal is seen to cut across a certain number of squares. Naturally therefore, the question arises whether the number of squares cut by a diagonal can be predicted when just the length and the breadth of the rectangle are known. This enquiry is within the level of maturity of children to pursue. What children normally do is to fix different rectangles on a square ruled sheet, use a thin broomstick to place along the diagonal in each, count the number of squares cut by the diagonal and tabulate their findings.

This investigation is challenging as the correct solution does not emerge immediately. At first partially correct solutions are discovered and their inadequacies lead children ultimately to the comprehensive and full solution covering all cases. A case study of discovery in stages is presented below with illustrations (see Fig. 1).

When the number of units in length and breadth are consecutive numbers, children discover that the number of squares cut by a diagonal is equal to one less than *the sum of the number of units in length and breadth*, i.e. $(l+b-1)$

When the number of units in length is a multiple of the number of units in breadth, the number of squares cut by a diagonal is simply equal to the number of units in length, that is 1.

In a square, the number of unit squares cut in it by a diagonal is simply the number of units in the side of the square, that is, a (see Fig. 3).

When the number of units of length and breadth are even numbers, the pattern becomes tricky. Some children veer round to determination of the number that is to be subtracted from the sum of the number of units in length and breadth. Once children tabulate their findings and examine the relation of the number to be subtracted to the number of units in length and breadth, they are on the threshold of a very significant discovery (see fig. 4).

l length in natural number units	b breadth in natural number units	Number of squares cut by a diagonal	Number to be subtracted from $(l + b)$
4	3	6	1
3	2	4	1
5	4	8	1
4	2	4	2
6	3	6	3
8	4	8	4
2	2	2	2
3	3	3	3
6	4	8	2
12	8	16	4

The number to be subtracted turns out to be the HCF (the highest common factor) of the number of units in length and the number of units in breadth. Children experience the thrill when this idea appears before their mental horizon. And this is an instance of a *non-routine application of HCF.*

Investigational Spree

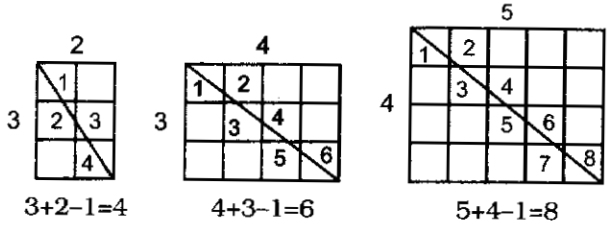


Fig. 1

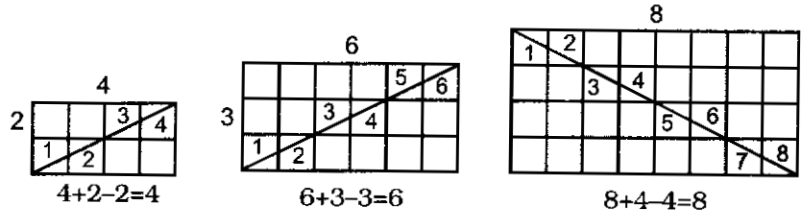


Fig. 2

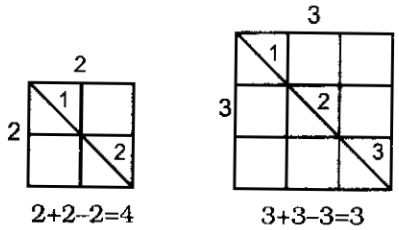


Fig. 3

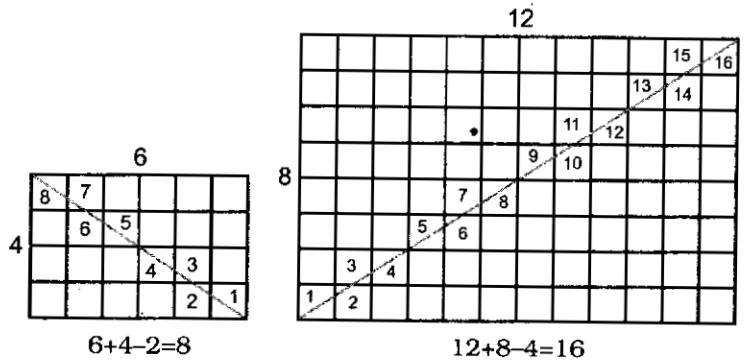


Fig. 4

E 4 – Narayana Pandita Game of Moves

1. Prologue

Children love action and get excited when the action has the element of suspense in it. If the action could give scope to exercise of rudimentary arithmetical skills, children grow in self-confidence and as a result develop a favourable attitude to mathematics right from primary years. A great gain indeed and children can be seen asking for more.

Of course we have to fish out suitable areas for providing rich mathematical experiences to children. A network of zing pathways engages children in the game of watching, discovering and predicting the pattern of accumulations at each row of junctions (nodes) while moving along the paths a collection of objects by observing the rule of half-splitting.

2. The game

Square ruled or check ruled sheets used in arithmetic classes can be made available to children. (Let the ruling be in centimetre or inch squares, preferably the latter). If the floor design at home or school is of square-rule, it can also be used. Three sheets would be enough for each child. Have a collection of 20 packets each having 10 seeds, 10 small coins or 10 small buttons, ready for use in this game of moving.

Cut out a 6 × 6 square from a square ruled sheet and cut it along one of its diagonals (lines joining opposite corners) to get two triangles. Paste on a cardboard one of the triangles (as shown below) with its square corner at the top and its base in the same direction as the bottom edge of the cardboard. This is just for convenience, to start with (see Fig. 1a).

Ask children to study the network of pathways seen in the triangle. Two downward paths emerge from each junction (or node). So the pathways can be set up zigzag. There are rows of junctions (or nodes). The number of junctions increases by one for each successive downward row from the top (see Fig. 1b).

Ask children to count and keep ready collections of seeds (in powers of two): 2 (first power of 2), 4 (second power of 2), 8 (third power of 2), 16, 32 and 64. (*Note:* Count of equal factors of a power number gives the index of the power: e.g. $2 \times 2 \times 2 \times 2 = 16$. This is written as $2^4 = 16$. 16 is a power of 2 and 4 is the index of the power). Let the children use the collections in turns, starting with. The rule of the game is simple and interesting. At each junction half-split the collection and move the halves along the two pathways going downward from the junction. The movement stops when half-splitting cannot be done further and the row of junctions at which this happens is considered the *terminal* or *final row.* The terminal row (indicated by arrows) changes according to the number of seeds with which movement is commenced. Ask children to record the number of seeds at the start, the index of the power of two it represents, the pattern of junction numbers in the terminal row in each case. What children would be experiencing on correct performance is pictured in Fig. 2.

The table of findings that children would draw up will be as follows:

Number of seeds at the start	Index of the power of 2 represented by the number	The sequence of junction numbers in the terminal row
2	1	11
4	2	121
8	3	1331
16	4	14641 and so on

Some questions will naturally crop up in the minds of children and in finding answers to those questions lies the exciting part of this *movement game.*

The questions are:

(1) Once the starting number of the collection of objects (restricted to powers of two) is given, which row becomes the terminal row ?

(2) How can the sequence of junction numbers in the

terminal row be predicted?

(3) Can the sequence of junction numbers in the terminal row be predicted for *any* power of two as the starting number?

(4) Can the sequence of junction numbers in *any* row be predicted for *any* power of two as starting number?

Children can be seen to discover, with a little hint if necessary, that

(1) The number of the terminal row is the *same* as the index of the power of two with which the movement starts.

(2) To get the sequence of junction numbers in the terminal row, start with 1 and write down numbers got by consecutive pairing and adding of numbers in the terminal row of the previous game of movement (see Fig. 3 a).

These rows of numbers are *basic* and if remembered equips one with a rich mathematical fare. The first basic row consists of 1 and 1. The second basic row consists of 1, 2 and 1. The third basic row consists of 1, 3, 3 and 1. And so on. This triangular array of numbers (giving the junction numbers of terminal rows) goes by the name of Pascal, a great French mathematician of the 17th century. But Narayana Pandita of 14th century mentions this in his *Ganita Kaumudi*. Hence the movement game is named after the latter.

(3) The sequence of junction numbers at any row can be seen to be simply multiples of numbers of the corresponding basic row. An example is given below (see Fig. 3 b).

> 8 is the *third* power of 2. The sequence of numbers in the *first* row consists 4 and 4. That is 4 × l, 4 × 1. 4 is the second power of 2. (2 got as 3 – 1). The sequence of number in the *second* row consists of 2, 4 and 2. That is 2 × 1, 2 × 2, 2 × 1, and 2 is the *first* power of 2 (1 got as 3 – 2). And so on.

The multiple is a power of two and the index of the power is got by subtracting the number of the row (at which the sequence of junction numbers is to be predicted) from the index of the power of two with which the movement game starts.

(4) The most challenging and rewarding experience is to reel off the numbers of any specific row in the triangular array of basic numbers, without extension row by row. The first thing that children notice is that first number in any row of basic numbers is *always* 1 and the next number in the row is the same as the (ranking) number of the two (from the top).

Throw this hint to children that they should try relating the powers in descending order from a certain stage and the basic numbers of the row at that stage. The pattern is rather deep and some children succeed in ferreting it out fast. They discover the need to associate the ranks of numbers in the row as they come to be fixed one by one. An illustration is given below:

Power	Index	Sequence of numbers in terminal pattern
2	1	11
4	2	121
8	3	1331
16	4	14641 and so on

3. Epilogue

These are not curio numbers to be played with and forgotten. They are profound and powerful. They make their appearance in binomial theorem, finite differences, probability theory, combinatorics, etc. Children exposed to these numbers in primary years would certainly be richer.

Narayana Pandita Game of Movement

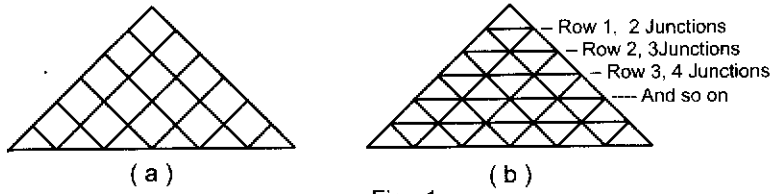


Fig. 1

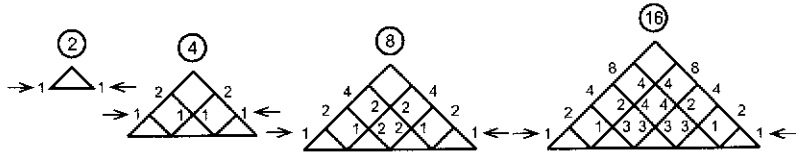
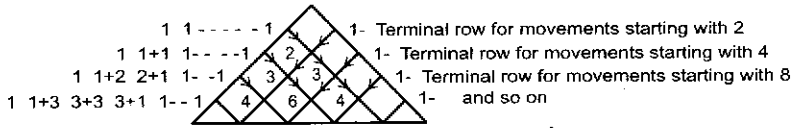


Fig. 2


1 1 - - - - - 1
1- Terminal row for movements starting with 2
1 1+1 1- - - -1
1- Terminal row for movements starting with 4
1 1+2 2+1 1- -1
1- Terminal row for movements starting with 8
1 1+3 3+3 3+1 1- - 1
1- and so on
(a)

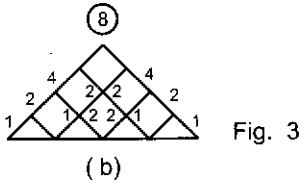


Fig. 3

(b)

To fix the numbers in the 4th row

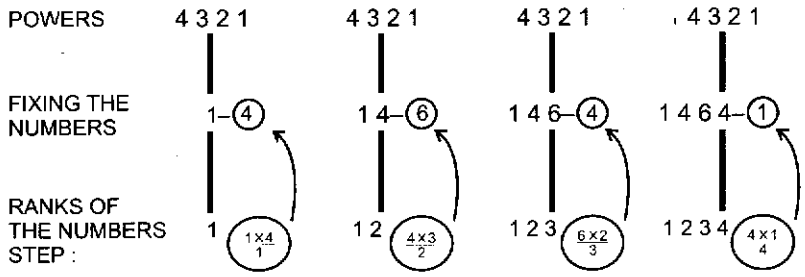


Fig. 4

E 5 – A Maths Project with Fruits

1. Prologue

When children go along with their parents to buy fruits, their attention gets naturally drawn to the piles of oranges and mangoes in pyramidal shapes. Some pyramids have triangular bases and some square bases. Sometimes rectangular bases are also seen with the pile in each, tapering to a single row at the top and the shapes are wedge like. These arrangements excite the curiosity of children and if only parents know how to sustain this curiosity by engaging children in a worthwhile project, children would get a fine opportunity to see mathematics at work.

2. Building triangular pyramids

It would help a lot if children are allowed to build pyramids with lemon fruit whenever bought in bulk at home to make pickles and discover the relation that the number of objects upto and including a layer from the top bears to the number of layers from the top. Ensure that children recognise first that the number of objects in each layer is triangular (1, 1 + 2, 1 + 2 + 3, 1 + 2 + 3 + 4, etc.) and the number of layers is the same as the number of objects in the outermost edge, if the base is triangular (or square). Children find that once a layer is fixed, the gaps or crevices among the objects in that layer show up the placements for objects to make the next top layer.

Ask children to tabulate their findings instead of setting up successively separate piles in different places, it is easier to build up piles in succession in the same place and make the recordings.

Number of layers from the top	Number of objects in these layers
1	1
2	4
3	10
4	20
5	35
6	56

The next question that would be of interest is to find out if there could be a rule or a formula to give the number of objects knowing the number of layers. Very few fruit sellers can give the number of the fruit in a pyramidal pile they have themselves built. So children see the power of mathematical thinking.

3. Search for formula

Let us see how a middle school child can be helped to find the rule. The child should be familiar with factorisation. Suggest to the child to factorise each of the numbers in the second column of the table and look for link with the corresponding number in the first column (see Fig. 1).

A sample of children's work that led to the discovery is given here.

Col. I	Col. II	Factorisation		
		Stage I	Stage II	Stage III
1	1	1	1	$\frac{1 \times 2 \times 3}{6}$
2	4	2×2	2×2	$\frac{2 \times 3 \times 4}{6}$
3	10	2×5	$\frac{2 \times 3 \times 5}{3}$	$\frac{3 \times 4 \times 5}{6}$
4	20	4×5	4×5	$\frac{4 \times 5 \times 6}{6}$
5	35	5×7	5×7	$\frac{5 \times 6 \times 7}{6}$
6	56	7×8	$\frac{6 \times 7 \times 8}{6}$	$\frac{6 \times 7 \times 8}{6}$

A beautiful pattern emerges giving the rule for finding the number of objects in a pyramidal pile with triangular base.

If the number of layers from the top is known, take the next two (higher) consecutive numbers and find the product of the three numbers. A sixth of the product gives the number of objects in the layers. Using symbols, the formula is easily struck. If n is the number of layers from the top of a triangular pyramid, the number of objects in n layers is simply

$$\frac{1n}{6}\ (n+1)\ (n+2)$$

4. Square pyramids

Ask children to build pyramidal piles with square bases and find the rule by tabulating their observations as before. By factorisation, children find that if n represents the number of layers from the top in a square pyramid, N the number of objects in these n layers then N is given by

$$N = \frac{1n}{6}(n+1)(2n+1)$$

5. Rectangular pyramids

A more challenging project is to have wedge like piles on a rectangular base and find the rule for the number of objects in terms of *n*, the number of layers (which is the same as the number of objects in the width-wise row of bottom-most layer) and d the difference between the number of objects in the outer most lengthwise row and widthwise row of the nth layer. As for instance a rectangular base pile can have layers having 7×4, 6×3, 5×2 and 4×1 objects. Here $d = 7-4 = 3$ and $n = 4$.

If need be, throw the hint that the square base pile is only a special case of the wedge like pile. Children can be seen to have a rewarding encounter and they come out with triumphant joy to announce that

$$N = \frac{n(n+1)\ (2n+1\ +\ 3d)}{6}$$

This formula reinforces their understanding as $d = 0$ makes it the formula for the square base pyramidal pile. Let children

know, incidentally, that d is a parameter, as it changes only from pile to pile and not in the same pile.

6. Project

Children can also be set the task of finding the number of objects in successive piles.

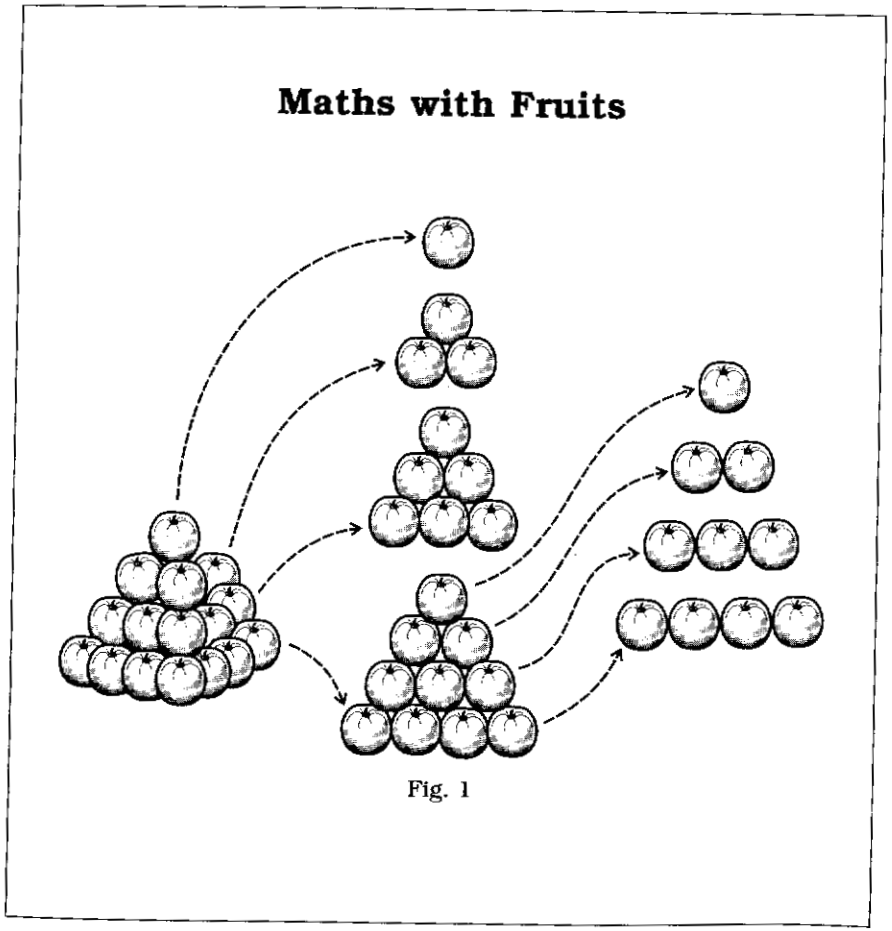


Fig. 1

E6 – Exciting Geometric Connections

1. Prologue

Children get excited when they discover unexpected connections. Mathematics offers plenty of scope for this kind of experience. Once children get excited, they look for more excitement and develop a favourable attitude towards mathematics. An interested teacher can easily win the adoration of children, if he cares to provide such opportunities while introducing a mathematical topic. An instance is cited here.

2. Investigation

Square ruled sheets are easy to get. Ask a child to cut out (i) strips having unit squares 1, 2, 3 and so on each in a row and then (ii) slips of squares of sizes 1×1, 2×2, 3×3 and so on.

(1) Set up the investigative task of picking up two strips and spotting out the third strip which has as many unit squares as the two strips together have. In cases where there are no concrete strips available, the child can suggest the existence of such a strip (see Fig.1).

(2) Next let the child extend the investigation to squares. A surprise awaits him now.

When the existence on the strip with as many unit squares as the unit squares contained in any two strips taken together can be visualised, the experience with square slips is different. Given any two square slips, it is not always possible to get a third square slip having as many unit squares as the unit squares in the two square slips taken together. The child can be encouraged to list cases where the above mentioned relation holds good such as (3,4,5), (6, 8, 10), (5, 12, 13) and so on (see Fig. 2).

3. Squares and triangles

Many children believe that with any three straight sticks (line segments) a triangle can be built. Ask children to build triangles with the lengths of any three strips in this investigation. They can discover that the *sum of two side lengths should necessarily be greater than the third side-length for a triangle to be formed.*

Even an animal knows that to reach one spot from the other, it should go straight and not through a third spot (forming a triangle with two spots), as the distance would be longer, better children understand the situation this way. While forming triangles, children should notice the formation of acute, obtuse and right triangles. *Pose the question of predicting the kind of triangle from the side-lengths.* Children find it challenging and obscure to start with, though they do see that the greatest angle is opposite to the greatest side and vice versa in a triangle.

The children may ask Why do we not build triangles with square sides? If so, well and good and flag them off to try and find out. If not, suggest them to do it. They should first ascertain which squares to choose by examining the lengths of square-sides (see Fig. 3).

When a square has *as many* unit squares *as* the unit squares in the other two squares taken together, the triangle formed by the two square sides is right angled (see Fig. 3b). Where a square has *more* unit squares than the unit squares in the other two squares taken together, the triangle formed by the two square sides is obtuse angled (see Fig. 3a). In the case of *less* unit squares, the triangle is acute angled (see fig. 3c).

4. Epilogue

This opens the stage for the study of right angled triangles with measures of sides given in real numbers, leading therefrom to the Pythagorean property of the right angled triangle in full. It is an exciting connection with numbers, squares and triangles to discover and enjoy!

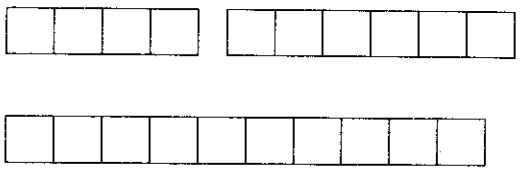


Fig. 1

Exciting Geometric Connections

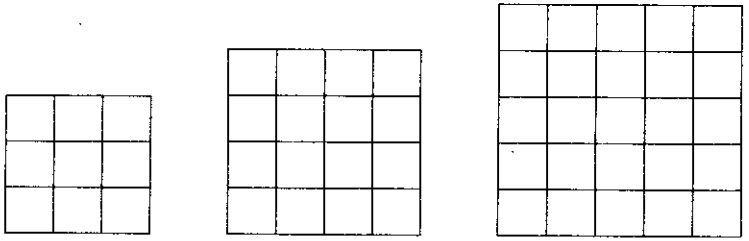


Fig. 2

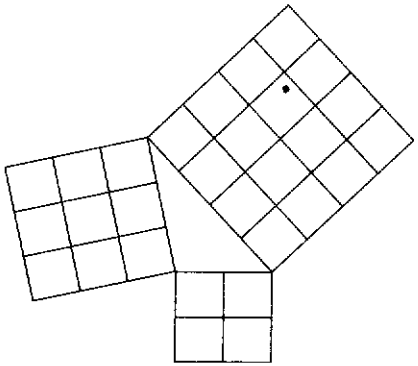


Fig. 3a

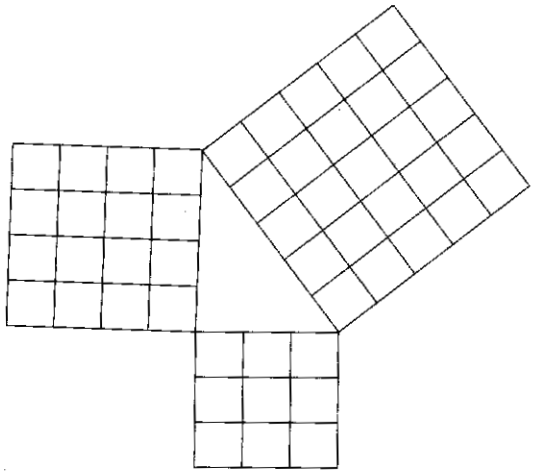


Fig. 3b

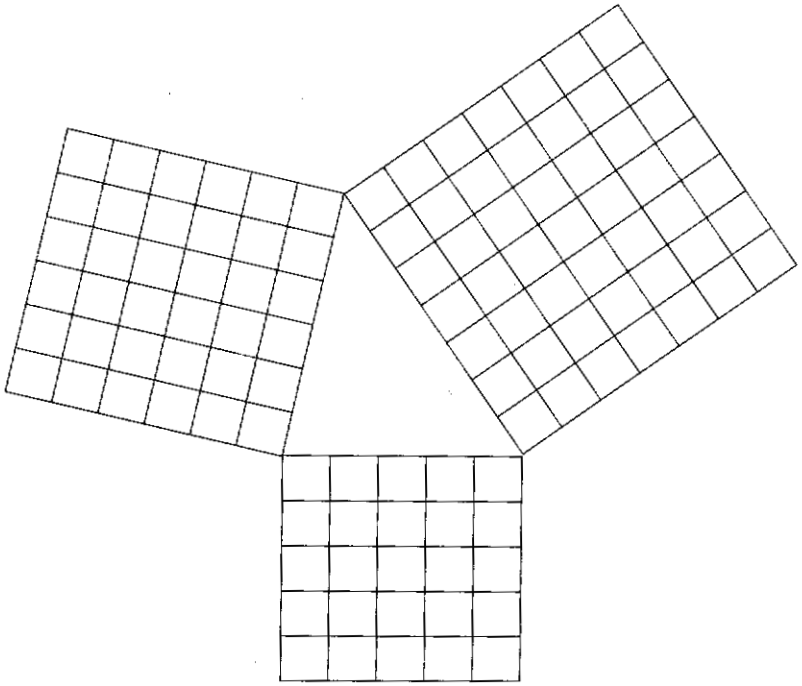


Fig. 3c